1. Is it possible to construct a square whose vertices are all lattice points? If so, provide an example. If not, give a proof that demonstrates why it's not possible.

Assume one of the points A of the square is at the origin (0, 0)

Let one sides of the square be a vector $\vec{v} = (a, b)$ (where a and be are integers and not 0)

One vertex of the square, B, will be B = (a, b)

A 90º rotation of B results in (-b, a), which is still a lattice point, but is the other side of the square

Third vertex C is (a-b, b+a)

Fourth vertex D is (-b, a)

All four vertices are (0, 0), (a, b), (a-b, b+a), (-b, a), which are all integer coordinates

Verifying it's still a square:

Adjacent sides (a, b) and (-b, a) both have length $a^2 + b^2$, , and dot product is $(a, b) \cdot (-b, a) = -ab + ab = 0$

Adjacent sides being equal length and perpendicular means that ABCD is square.

2. Is it possible to construct an equilateral triangle whose vertices are all lattice points? If so, provide an example. If not, give a proof that demonstrates why it's not possible.

Assume one vertex of the triangle, A, is at the origin $(0, 0)$

Let a second vertex be $B = (a, b)$, where a and b are integers and not both zero

To form an equilateral triangle, the third vertex C must be obtained by rotating $\vec{AB} = (a, b)$ by 60°

A 60° rotation matrix is

$$R_{60} = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Applying this rotation to (a, b) gives

$$C = \left(\frac{a - b\sqrt{3}}{2}, \frac{a\sqrt{3} + b}{2} \right)$$

For C to be a lattice point, both coordinates must be integers

However, since $\sqrt{3}$ is irrational, the expressions

$$\frac{a - b\sqrt{3}}{2}, \quad \frac{a\sqrt{3} + b}{2}$$

cannot both be integers unless $a = b = 0$, which gives a degenerate triangle

Therefore, it is impossible to construct a non-degenerate equilateral triangle whose vertices are all lattice points.

3. Is it possible to construct a regular (equiangular and equilateral) pentagon whose vertices are all lattice points? If so, provide an example. If not, give a proof that demonstrates why it's not possible.

Assume one vertex of the pentagon is at the origin $(0, 0)$

Let a second vertex be $B = (a, b)$, where a and b are integers

A regular pentagon requires successive rotations of $\vec{AB}$ by $72°$

We can get the cosine and sine of $72°$ by expressing $72°$ as $90° - 18°$
$\sin 72° = \sin(90° - 18°)$
From the trigonometric identity $\sin(90 - \theta) = \cos\theta$
Therefore, $\sin(90° - 18°) = \cos 18°$
$\cos\theta = \sqrt{1 - \sin^2\theta}$, from the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$
We can get $\sin 18°$ by
The cosine and sine of $72°$ are

$$\cos 72° = \frac{\sqrt{5} - 1}{4}, \quad \sin 72° = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Applying a $72°$ rotation to (a, b) produces coordinates involving $\sqrt{5}$

Thus, each new vertex has coordinates that are linear combinations of a and b with irrational coefficients

Since lattice points require integer coordinates, and $\sqrt{5}$ is irrational, no such rotated point can lie on the integer lattice unless the pentagon is degenerate

It is impossible to construct a regular pentagon whose vertices are all lattice points

4. Compute, with justification, the following quantities:

(a) $\tan\frac{\pi}{3}$

(b) $\tan\frac{\pi}{4}$

(c) $\tan\frac{\pi}{5}$

(a) $\tan\frac{\pi}{3}$

From special angle values,

$$\tan\frac{\pi}{3} = \sqrt{3}$$

(b) $\tan\frac{\pi}{4}$

In a $45°$-$45°$-$90°$ triangle, opposite and adjacent sides are equal, so

$$\tan\frac{\pi}{4} = 1$$

(c) $\tan\frac{\pi}{5}$

Using the identity

$$\tan 36^\circ = \sqrt{5-2\sqrt{5}},$$

and noting that $\frac{\pi}{5} = 36^\circ$, we obtain

$$\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}}$$

This value is irrational, consistent with the fact that regular pentagons require irrational trigonometric values

5. Use Exercise 4 to show that it is not possible to construct a regular lattice octagon (8-gon).

Let the side length of the regular octagon be s. By Pick's Theorem, the area of any polygon with integer coordinates is either an integer or a half-integer, which means the area must be a rational number

The area of a regular n-gon is given by $A = \frac{n}{4}s^2\cot\left(\frac{\pi}{n}\right)$. For an octagon ($n = 8$), $A = 2s^2\cot\left(\frac{\pi}{8}\right)$

From Exercise 4(b), we know $\tan\left(\frac{\pi}{4}\right) = 1$. Using the double angle formula for tangent:

$$\tan\left(\frac{\pi}{4}\right) = \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} \implies 1 = \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$$

Solving for $\tan\left(\frac{\pi}{8}\right)$ gives $\tan^2\left(\frac{\pi}{8}\right) + 2\tan\left(\frac{\pi}{8}\right) - 1 = 0$, so $\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$ (taking the positive root)

Thus, $\cot\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$.

The area of the octagon is therefore $A = 2s^2(\sqrt{2}+1) = 2s^2 + 2s^2\sqrt{2}$.

Since the vertices are lattice points, the distance squared between any two adjacent points, $s^2 = (\Delta x)^2 + (\Delta y)^2$, must be an integer. This makes $2s^2\sqrt{2}$ an irrational number, so the total area A is irrational. This contradicts Pick's Theorem, meaning a regular lattice octagon cannot exist

6. Show that if α is a lattice angle and if the measure of α is not equal to $\pi/2$ or an odd integer multiple of $\pi/2$, then the tangent of α is a rational number.

Let α be an angle formed by three lattice points A, B, and C, with the vertex at B. We can translate the points such that B is at the origin $(0,0)$. Let A have coordinates (x_1, y_1) and C have coordinates (x_2, y_2). Since A, B, C are lattice points, $x_1, y_1, x_2, y_2 \in \mathbb{Z}$

Let $\mathbf{u} = \langle x_1, y_1\rangle$ and $\mathbf{v} = \langle x_2, y_2\rangle$. The angle α is the angle between $\mathbf{u}$ and $\mathbf{v}$

Using the dot product and the 2D cross product magnitude:

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{x_1x_2 + y_1y_2}{|\mathbf{u}||\mathbf{v}|}$$

$$\sin\alpha = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|} = \frac{|x_1y_2 - x_2y_1|}{|\mathbf{u}||\mathbf{v}|}$$

The tangent of α is the ratio of sine to cosine:

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{|x_1 y_2 - x_2 y_1|}{x_1 x_2 + y_1 y_2}$$

Since the coordinates are all integers, both the numerator and denominator are integers. As long as $\alpha \neq \frac{\pi}{2} + k\pi$ (which implies $\cos \alpha \neq 0$ and $x_1 x_2 + y_1 y_2 \neq 0$), the denominator is a non-zero integer. Thus, $\tan \alpha$ evaluates to a rational number

> 7. Make a conjecture about the positive integers n for which it is possible construct a regular lattice n-gon. Although you do not need to provide a formal proof of your conjecture here, you should provide sufficient justification and reasoning (beyond simply citing the exercises that you have already solved above) to indicate why you believe your conjecture is valid.

Conjecture: The only positive integer $n \geq 3$ for which it is possible to construct a regular lattice n-gon is $n = 4$ (a square)

Justification: For a regular n-gon to be a lattice polygon, its interior angle $\alpha = \frac{n-2}{n}\pi$ must either have a rational tangent or be exactly $\pi/2$ (as established in Exercise 6)

For $n = 3$, the interior angle is $\pi/3$, and $\tan(\pi/3) = \sqrt{3} \notin \mathbb{Q}$ For $n = 4$, the angle is $\pi/2$, which is achievable (e.g., vertices $(0,0), (1,0), (1,1), (0,1)$ form a lattice square). For $n = 6$, the interior angle is $2\pi/3$, and $\tan(2\pi/3) = -\sqrt{3} \notin \mathbb{Q}$

> 8. For which positive integers n is it possible to construct an *equilateral* (but not necessarily regular) lattice n-gon?

It is possible to construct an equilateral lattice n-gon if and only if n is an even integer such that $n \geq 4$

To form an equilateral polygon, the sequence of edge vectors $\mathbf{v}_i = \langle x_i, y_i \rangle$ must all share the exact same squared length $L^2 = x_i^2 + y_i^2$, and must sum to zero to form a closed loop: $\sum x_i = 0$ and $\sum y_i = 0$

Because L^2 is an integer, we can analyze the modulo 2 parity of the coordinates. We know $x_i^2 + y_i^2 \equiv x_i + y_i \pmod 2$. Summing this parity over all n edges gives $\sum(x_i + y_i) \equiv nL^2 \pmod 2$. Because the sum of the edge components is 0, we have $0 \equiv nL^2 \pmod 2$. If n is odd, L^2 is forced to be even, meaning $x_i \equiv y_i \pmod 2$. By continuously dividing the polygon by 2 (shrinking it), we eventually reduce the problem to the case where all x_i and y_i are odd. This results in $L^2 \equiv x_i^2 + y_i^2 \equiv 1 + 1 = 2 \pmod 4$

Since $\sum x_i = 0$ and all remaining x_i components are odd, you must add an *even* number of odd integers to get to zero. Therefore, n must be even

For any even integer $n \geq 4$, an equilateral polygon can be easily constructed by selecting a set of distinct lattice vectors of identical length L, pairing them with their negations (e.g., $\mathbf{v}$ and $-\mathbf{v}$), and arranging them symmetrically head-to-tail to form a convex figure

> 9. Is it possible to construct a lattice square whose area is not a perfect square? If so, provide an example. If not, prove it.

Yes, it is possible!!!

Consider a square formed by the lattice vertices $(0,0),(2,-1),(3,1)$, and $(1,2)$

The vectors defining two adjacent edges originating from $(2,-1)$ are: $\mathbf{v}_1 = \langle 0-2, 0-(-1)\rangle = \langle -2, 1\rangle$ $\mathbf{v}_2 = \langle 3-2, 1-(-1)\rangle = \langle 1, 2\rangle$

These adjacent edge vectors have a dot product of $\mathbf{v}_1 \cdot \mathbf{v}_2 = (-2)(1) + (1)(2) = 0$, confirming the angle between them is exactly $90°$

The side length s of the square is the magnitude of $\mathbf{v}_2$: $s = \sqrt{1^2 + 2^2} = \sqrt{5}$

The area of the square is $s^2 = (\sqrt{5})^2 = 5$. Since 5 is an integer but is not a perfect square, this serves as a valid example

10. Is it possible to construct a lattice square whose area is not an integer? If so, provide an example. If not, prove it.

No, it is not possible.

Proof: Let the vertices of a lattice square be placed on the integer lattice $\mathbb{Z}^2$. The side length s is the Euclidean distance between any two adjacent vertices, say (x_1, y_1) and (x_2, y_2)

The area of the square is given by the side length squared: Area $= s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Since the vertices are lattice points, their coordinates x_1, y_1, x_2, y_2 are all integers. The differences $(x_2 - x_1)$ and $(y_2 - y_1)$ must also be integers because the set of integers is closed under subtraction

Squaring these differences produces integers, and summing them produces an integer (since the set of integers is closed under multiplication and addition). Therefore, the area $(x_2 - x_1)^2 + (y_2 - y_1)^2$ will always evaluate to an integer. It is algebraically impossible to construct a lattice square with a non-integer area

11. Show that there exists a lattice square with area n, where n is a positive integer, if and only if there exist non-negative integers a and b such that

$$n = a^2 + b^2$$

($\Rightarrow$) Assume there exists a lattice square with area n

Place one vertex of the square at the origin and let one side of the square be the vector

$$\vec{v} = (a, b), \quad a, b \in \mathbb{Z}$$

A side perpendicular to $\vec{v}$ with the same length is obtained by a $90°$ rotation:

$$\vec{w} = (-b, a)$$

The area of the square equals the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$:

$$\text{Area} = |\vec{v} \times \vec{w}| = a^2 + b^2$$

Since the square has area n, we get

$$n = a^2 + b^2$$

($\Leftarrow$) Now assume

$$n = a^2 + b^2$$

for some non-negative integers a, b

Consider the vectors

$$\vec{v} = (a, b), \quad \vec{w} = (-b, a)$$

By the Pythagorean Theorem,

$$|\vec{v}|^2 = a^2 + b^2 = |\vec{w}|^2,$$

so the two sides have equal length

Their dot product is

$$\vec{v} \cdot \vec{w} = a(-b) + b(a) = 0,$$

so the sides are perpendicular.

Thus these vectors form a lattice square whose area is

$$a^2 + b^2 = n$$

Therefore, a lattice square of area n exists *if and only if*

$$n = a^2 + b^2$$

12. Is it possible to construct a lattice triangle whose area is not an integer? If so, provide an example. If not, prove it.

Yes, it is possible to construct a lattice triangle whose area is not an integer

Let's construct a simple right-angled triangle with its vertices on the axes

Let the first vertex A be at the origin $(0, 0)$

Let the second vertex B be at $(1, 0)$

Let the third vertex C be at $(0, 1)$

All three vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ have integer coordinates, making them valid lattice points

The length of the base (from A to B) is 1

The height of the triangle (from A to C) is 1

The area of a triangle is given by the formula $\frac{1}{2} \cdot \text{base} \cdot \text{height}$

Area $= \frac{1}{2} \cdot 1 \cdot 1 = 0.5$

Since 0.5 is not an integer, we have successfully constructed a lattice triangle whose area is not an integer

13. Construct (at least) 5 lattice polygons with different areas. Keep in mind that the polygons do not need to be convex! Find the area of each polygon. What do you observe? Make a conjecture about the possible values of the area of a lattice polygon based on your computations in this problem. For example, do you think it's possible to achieve all integer areas? All rational areas? All real areas?

Let's construct 5 different lattice polygons using simple shapes and calculate their areas.

Polygon 1 (Triangle): Vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Base $= 1$, Height $= 1$. Area $= \frac{1}{2} \cdot 1 \cdot 1 = 0.5$

Polygon 2 (Square): Vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Side length $= 1$. Area $= 1 \cdot 1 = 1$

Polygon 3 (Triangle): Vertices at $(0,0)$, $(3,0)$, and $(0,1)$. Base $= 3$, Height $= 1$. Area $= \frac{1}{2} \cdot 3 \cdot 1 = 1.5$

Polygon 4 (Rectangle): Vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$. Length $= 2$, Height $= 1$. Area $= 2 \cdot 1 = 2$.

Polygon 5 (Triangle): Vertices at $(0,0)$, $(5,0)$, and $(0,1)$. Base $= 5$, Height $= 1$. Area $= \frac{1}{2} \cdot 5 \cdot 1 = 2.5$

Observations and Conjecture:

Based on the calculated areas (0.5, 1, 1.5, 2, 2.5), I observe that all the areas are exact multiples of 0.5 (they are either integers or half-integers)

Conjecture: The area of any lattice polygon is always a multiple of 0.5.

Therefore, it is possible to achieve all positive integer areas. For example, a rectangle with vertices $(0,0),(n,0),(n,1),(0,1)$ will always have an integer area of n

It is not possible to achieve all rational areas. Because the area must be a multiple of 0.5, a rational area like $\frac{1}{3}$ or $\frac{3}{4}$ cannot be constructed

It is not possible to achieve all real areas. Irrational areas like $\sqrt{2}$ or π are impossible to construct using only integer coordinates

14. Let T be a lattice triangle with $I(T) = 0$. What are the possible values of $B(T)$? Prove your result.

(Solution inspired by Palma's, especially the Pick's theorem aspect)

By Pick's Theorem,

$$\text{Area}(T) = I(T) + \frac{B(T)}{2} - 1$$

Since $I(T) = 0$, this simplifies to

$$\text{Area}(T) = \frac{B(T)}{2} - 1$$

Because T is a non-degenerate lattice triangle, its area is positive:

$$\frac{B(T)}{2} - 1 > 0$$

Hence,

$$B(T) > 2$$

Also, the area of any lattice triangle is a multiple of $\frac{1}{2}$, so

$$\frac{B(T)}{2} - 1 = \frac{k}{2} \quad \text{for some integer } k \geq 1$$

Solving gives

$$B(T) = k + 2$$

Thus $B(T)$ can be any integer ≥ 3

However, a triangle must have at least three boundary lattice points (its vertices), and it is possible to construct examples realizing every integer $B \geq 3$

Therefore, the possible values are

$$\boxed{B(T) = 3, 4, 5, 6, \ldots}$$

> 15. Make a conjecture about the possible values of B for a lattice triangle T with $I(T) = 1$. Provide justification and reasoning.

Again using Pick's Theorem,

$$\text{Area}(T) = I(T) + \frac{B}{2} - 1$$

With $I(T) = 1$, this becomes

$$\text{Area}(T) = \frac{B}{2}$$

The area is a half-integer multiple determined directly by B

Since the area of a lattice triangle must be a multiple of $\frac{1}{2}$, every integer value of B that produces a triangle area is a candidate

Experimenting with small lattice triangles having exactly one interior lattice point shows that

$$B \geq 3$$

Geometrically, increasing B corresponds to stretching the triangle along lattice directions while preserving exactly one interior lattice point

Which suggests the conjecture:

$$\boxed{\text{For } I(T) = 1, \text{ all integers } B \geq 3 \text{ are possible}}$$

> 16. Construct (at least) 5 different non-congruent primitive lattice triangles, and find their area. Make a conjecture about the value of the area of a primitive lattice triangle.

A primitive lattice triangle is a triangle whose only lattice points are its three vertices (meaning it has 0 interior lattice points and 0 boundary lattice points between the vertices)

Non-congruent triangles can be constructed by keeping a base of length 1 on the x-axis and placing the third vertex at a height of 1, shifting it further along the x-axis each time

Triangle 1: Vertices at (0, 0), (1, 0), and (0, 1). Sides are 1, 1, and $\sqrt{2}$

Triangle 2: Vertices at (0, 0), (1, 0), and (2, 1). Sides are 1, $\sqrt{2}$, and $\sqrt{5}$

Triangle 3: Vertices at (0, 0), (1, 0), and (3, 1). Sides are 1, $\sqrt{5}$, and $\sqrt{10}$

Triangle 4: Vertices at (0, 0), (1, 0), and (4, 1). Sides are 1, $\sqrt{10}$, and $\sqrt{17}$

Triangle 5: Vertices at (0, 0), (1, 0), and (5, 1). Sides are 1, $\sqrt{17}$, and $\sqrt{26}$

Because all five triangles have a base of 1 (the segment from (0,0) to (1,0)) and a height of 1 (the y-coordinate of the third vertex), their areas are all identical

Area $= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 1 \cdot 1 = 0.5$

Conjecture: Based on these computations, I conjecture that the area of any primitive lattice triangle is exactly 0.5

17. Construct several (at least 5) different lattice polygons that contain 4 boundary lattice points and 6 interior lattice points. Keep in mind that the polygons do not need to be convex! Find the area of each polygon. What do you observe? Make a conjecture based on your observations in this exercise.

To guarantee exactly 4 boundary points and 6 interior points, we can construct triangles with a base of length 2 (which gives 3 boundary points: the two ends and one in the middle) and a height of 7. If we pick the third vertex carefully so that the other two legs hit no integer coordinates, we will have exactly 4 boundary points

Polygon 1: Triangle with vertices (0, 0), (2, 0), and (1, 7) Boundary points: (0, 0), (1, 0), (2, 0), and (1, 7) (Total: 4) Area $= \frac{1}{2} \cdot 2 \cdot 7 = 7$

Polygon 2: Triangle with vertices (0, 0), (2, 0), and (3, 7) Boundary points: (0, 0), (1, 0), (2, 0), and (3, 7) (Total: 4) Area $= \frac{1}{2} \cdot 2 \cdot 7 = 7$

Polygon 3: Triangle with vertices (0, 0), (2, 0), and (5, 7). Boundary points: (0, 0), (1, 0), (2, 0), and (5, 7). (Total: 4) Area $= \frac{1}{2} \cdot 2 \cdot 7 = 7$

Polygon 4: Triangle with vertices (0, 0), (0, 2), and (7, 3) Boundary points: (0, 0), (0, 1), (0, 2), and (7, 3). (Total: 4) Area $= \frac{1}{2} \cdot 2 \cdot 7 = 7$

Polygon 5: Triangle with vertices (0, 0), (0, 2), and (7, 5). Boundary points: (0, 0), (0, 1), (0, 2), and (7, 5). (Total: 4) Area $= \frac{1}{2} \cdot 2 \cdot 7 = 7$

Observation: Despite having different shapes and orientations, every single polygon constructed with exactly 4 boundary points and 6 interior points has an area of exactly 7

The area of a lattice polygon is completely determined by the number of its interior lattice points and boundary lattice points. Specifically, based on these exercises, it seems to follow the formula: Area $=$ Interior Points $+ \frac{\text{Boundary Points}}{2} - 1$

18. Find the area of each of the lattice polygons in Figure 1. Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B).

Screenshot 2026-03-13 at 15.07.58.png

Screenshot 2026-03-13 at 15.08.19.png

19. Construct 5 different lattice polygons. To keep this problem interesting, at least 3 of your polygons should be non-convex. All of your polygons should have at least 6 sides, and at least 10 boundary lattice points and at least 8 interior lattice points. For each of these 5 polygons, find the area, the number of lattice pointsinside the polygon, and the number of lattice points on the boundary of the polygon. Add this information to your table from Exercise 18.

1)

Screenshot 2026-02-13 at 15.13.34.png

2)

Screenshot 2026-02-13 at 15.11.55.png

3)

Screenshot 2026-02-13 at 15.08.50.png

4)

Screenshot 2026-02-13 at 15.06.58.png

5)

Screenshot 2026-02-13 at 15.03.51.png

Areas, interior lattice points, and boundary lattice points:

Screenshot 2026-02-13 at 17.15.51.png

20. Let P be the triangle with vertices $(0,0)$, $(3,1)$, and $(1,4)$. Find the area of P, the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon. Add this information to your table from Exercise 18.

First, we find the area of P using the Shoelace formula for vertices $(0,0)$, $(3,1)$, and $(1,4)$,

$$\text{Area}(P) = \frac{1}{2}|0(1-4) + 3(4-0) + 1(0-1)| = \frac{1}{2}|0 + 12 - 1| = \frac{11}{2} = 5.5$$

Next, we find the number of lattice points on the boundary, B, by summing the greatest common divisors of the coordinate differences for each edge,

$$B = \gcd(3-0, 1-0) + \gcd(|1-3|, 4-1) + \gcd(|0-1|, |0-4|)$$

$$B = \gcd(3,1) + \gcd(2,3) + \gcd(1,4) = 1 + 1 + 1 = 3$$

Now, using Pick's Theorem, we can solve for the number of interior lattice points, I,

$$\text{Area}(P) = I + \frac{B}{2} - 1$$

Substituting our known values,

$$\frac{11}{2} = I + \frac{3}{2} - 1$$

$$5.5 = I + 0.5 \implies I = 5$$

Which gives the final values for the table:

$$\boxed{\text{Area}(P) = 5.5, \quad I = 5, \quad B = 3}$$

21. Based on your work so far, make a conjecture about the area of a lattice polygon in the case when B (the number of lattice points on the boundary) is even and in the case when B is odd.

Coincidentally, none of the figures drawn above had an odd number of boundary points, so figure 5 has been modified to have 19 boundary points instead of 20:

Screenshot 2026-02-13 at 17.20.00.png

This new figure has B=19, I=10, and A = 18.5

Conjecture: The area of lattice polygons with an odd number of lattice boundary points is **not** an integer, but the area of lattice polygons with an even number of lattice boundary points is an integer.

22. Based on your work so far, conjecture a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon. Explain how you obtained your conjecture, and why you think it makes sense (including a proof or partial proof if you have ideas). Although you do not need to provide a formal proof here, you should provide sufficient justification and reasoning to indicate why you believe your conjecture is valid. Please do not try to find the formula online or in another reference–it's so much more fun and interesting if you discover the formula on your own! If you're not sure where to start, start with thinking about a linear relationship. If I increases by 1 and B stays fixed, what happens to the area? Similarly, if B increases by 1 and I stays fixed, what happens to the area? Use these observations to try to find an equation that relates A, B, and I.

Mathematical justification:

In both Exercise 19.3 and 19.5 the number of interior points (I) is exactly 10

Between these two cases, the number of boundary points (B) increases from 14 to 20, which is a net increase of 6

During this same change, the area increases from 16 to 19, which is a net increase of 3

Because an increase of 6 boundary points results in an area increase of 3, we can conclude that each boundary point contributes exactly 0.5 (or 1/2) to the total area

If we take Exercise 19.2 (A=16,I=12,B=10) and subtract the boundary contribution (0.5×10=5), we are left with a value of 11

If we take Exercise 19.3 (A=16,I=10,B=14) and subtract the boundary contribution (0.5×14=7), we are left with a value of 9

Comparing these two adjusted results shows that when I increases by 2 (from 10 to 12), the area increases by exactly 2(from 9 to 11). This 1:1 ratio proves that each interior point contributes exactly 1 full unit to the area

We now have a rough formula: A=I+0.5B+C, where C is a constant offset

Using Exercise 19.4 (A=15,I=8,B=16), we plug in the values: 15=8+0.5(16)+C. This simplifies to 15=8+8+C, which means 15=16+C

Solving for C gives us -1, giving the final formula $A = I + 1/2B - 1$

Logical justification:

Every interior point I is completely surrounded by the polygon, meaning all 360° of the area around that point is contained within the shape
An interior point occupies a full 1×1 square of the grid, it contributes exactly 1 unit to the total area.

Boundary points B that lie on a straight line are shared between the interior and the exterior of the polygon, and since exactly 180° (half of a full circle) of the space around such a point is inside the polygon, it contributes exactly 0.5 units to the area

The sum of the interior angles of any polygon with n vertices is always $(n-2) \times 180°$

Because 360° is a full point, the sum of the vertices contributes $\frac{n-2}{n}$ to the area, which can be rewritten as $\frac{n}{2} - 1$, which explains why the number of boundary points is divided by two and then reduced by one

23. Construct your own lattice polygon, and verify that the formula that you conjectured in Exercise 22 is satisfied. Your polygon should be at least somewhat interesting–for example, make it non-convex, and have at least 10 boundary lattice points and at least 8 interior lattice points.

Screenshot 2026-02-13 at 18.29.35.png

$B = 16, I = 9$

Conjectured area: $A = I + B/2 - 1 = 16$

Area: 16

The area of this polygon satisfies the formula conjectured in Exercise 22

24. Assuming that your conjecture in Exercise 22 is valid, make a conjecture about the possible values of the area of a lattice polygon. Can you construct a lattice polygon with area 5/2, for example? How about 5/3? Which real numbers are possible? Explain how you obtained your conjecture, and why you think it makes sense (including a proof or partial proof if you have ideas). Although you do not need to provide a formal proof here, you should provide sufficient justification and reasoning to indicate why you believe your conjecture is valid.

The area of any lattice polygon must be a multiple of 1/2, so the set of all possible areas (excluding null area 0) is $\{0.5, 1, 1.5, 2, 2.5, ...\}$

Any real number that is not a multiple of 1/2 (such as 5/3 or π) is impossible for a lattice polygon

5/2 is a possible area, since to construct such a polygon we can choose I = 0 and B = 7 to get an area of A = B/2 - 1 = 2.5

Since B, I, and 1 are all integers, the B/2 term can only lead to A having a whole number value or a value that ends with 0.5. Therefore, and area of 5/3 is impossible since no combinations of integers I, B, and 1 would result in a fraction with a denominator of 3

Any real number that cannot be expressed as $n/2$, where n is an integer, is an impossible lattice polygon area

Logically, this makes sense since I will always be an integer as you cannot have a fraction of a point inside a polygon, B/2 will either be a whole number if B is even or a number ending in 0.5 if B is odd. The integer 1 is not going to add a decimal to the the result, it will still end in a .0 or .5. Also, geometrically, any lattice polygon can be deconstructed into a series of triangles with an area of 1/2, so logically any lattice polygon has to be some sum of 0.5s

25. Does every lattice line segment have rational length? If so, prove it. If not, provide an example of a lattice line segment with non-rational length.

No, not every lattice line segment has a rational length

Consider a lattice line segment with endpoints at $A = (0, 0)$ and $B = (1, 1)$

Using the distance formula, the length d is: $d = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

Since $\sqrt{2}$ is an irrational number, this lattice line segment has a non-rational length

26. Show that the square of the length of any lattice line segment is an integer.

Let the endpoints of a lattice line segment be (x_1, y_1) and (x_2, y_2), where x_1, y_1, x_2, y_2 are all integers

The length L of the segment is given by: $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The square of the length is: $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Since the set of integers is closed under subtraction and multiplication, $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ must both be integers

The sum of two integers is also an integer, therefore L^2 is an integer

> 27. Let L be a line with rational slope in the plane. Show that if there is a lattice point on L, then the y-intercept of L is rational.

Let the line L have a rational slope $m = \frac{p}{q}$ (where p and q are integers and $q \neq 0$)

Let (x_0, y_0) be a lattice point on the line L, so x_0 and y_0 are integers

The equation of the line in point-slope form is: $y - y_0 = m(x - x_0)$

To find the y-intercept, we set $x = 0$ and solve for y: $y - y_0 = m(0 - x_0)$ $y = y_0 - m \cdot x_0$

Substituting the rational slope $m = \frac{p}{q}$: $y = y_0 - \frac{p}{q} \cdot x_0$ $y = \frac{q \cdot y_0 - p \cdot x_0}{q}$

Since p, q, x_0, and y_0 are all integers, both the numerator $(q \cdot y_0 - p \cdot x_0)$ and the denominator q are integers

A ratio of two integers is a rational number. Thus, the y-intercept is rational

> 28. Let L be a line with rational slope in the plane. Show that if there is one lattice point on L, then there are infinitely many lattice points on L.

Assume the slope of L is rational. Then we write $m = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$, $q \neq 0$

Suppose there exists one lattice point on L, e. g. $P_0 = (x_0, y_0) \in \mathbb{Z}^2$

Since L has slope $\dfrac{p}{q}$, a direction vector for L is $\vec{v} = (q, p)$

$\frac{\text{rise}}{\text{run}} = \frac{p}{q}$

Because p and q are integers, $\vec{v}$ is a lattice vector

Every point on L can be written as $P(t) = (x_0, y_0) + t(q, p), \quad t \in \mathbb{R}$

Choose $t = k$ where $k \in \mathbb{Z}$. Then $P(k) = (x_0 + kq, ; y_0 + kp)$

Since x_0, y_0, p, q, k are all integers, each coordinate of $P(k)$ is an integer, so $P(k) \in \mathbb{Z}^2$

Because there are infinitely many integers k, this produces infinitely many distinct lattice points on L

Therefore, if a line with rational slope contains one lattice point, it contains infinitely many lattice points

29. Let $p = (m, n)$ be a lattice point with $\gcd(m, n) = 1$. Show that there are no lattice points strictly between $0 = (0, 0)$ and p on the line segment $0p$.

Points on the line segment from $(0, 0)$ to (m, n) have the form $(tm, , tn)$, $\quad 0 \le t \le 1$

Suppose there exists a lattice point between the endpoints. Then there exists $0 < t < 1$ such that $(tm, tn) = (a, b)$ with $a, b \in \mathbb{Z}$

Thus $t = \frac{a}{m} = \frac{b}{n}$

From $a = tm$ and $b = tn$, we get $an = bm$

Since $\gcd(m, n) = 1$, m and n are relatively prime integers and so if $m \mid an$, then $m \mid a$

Therefore there must exist an integer k such that $a = km$

Substituting into $an = bm$ gives $kmn = bm$, so dividing by m (which is nonzero unless $p = (0, 0)$, excluded here for simplicity), $kn = b$

So $(a, b) = (km, kn) = k(m, n)$

But since the point lies between $(0, 0)$ and (m, n), we must have $0 < k < 1$

There doesn't exist an integer between 0 and 1

Thus no such integer k exists

Therefore, there are no lattice points strictly between $(0, 0)$ and (m, n) when $\gcd(m, n) = 1$

30. Show that if $p = (m, n)$ is a visible point on the lattice line L through the origin $(0, 0)$, then any lattice point on L is of the form (tm, tn) for some integer t.

By the definition of a visible point from the origin, the integers m and n share no common positive factors other than 1. In basic terms, this means the fraction $\frac{n}{m}$ is entirely reduced and in its simplest form.

The line L passes through the origin $(0, 0)$ and the point (m, n), giving it a slope of $\frac{n}{m}$

Let (x, y) be any other arbitrary lattice point on L, meaning x and y must both be integers

Because (x, y) lies on the exact same line originating from $(0, 0)$, the ratio of its coordinates must equal the line's slope: $\frac{y}{x} = \frac{n}{m}$

Since $\frac{n}{m}$ is a fully reduced fraction, the only mathematical way to create an equivalent integer fraction $\frac{y}{x}$ is to multiply both the numerator and the denominator of the reduced fraction by the exact same integer

Let's call this integer multiplier t.

Therefore, to keep the ratio equal, x must be m scaled by t, and y must be n scaled by t, giving us $x = tm$ and $y = tn$

Thus, any lattice point on L is of the form (tm, tn) for some integer t

31. Let m and n be nonnegative integers. Show that there are exactly $\gcd(m, n) - 1$ lattice points on the line segment between the origin and the point (m, n), not including the endpoints.

Let $d = \gcd(m, n)$ and write $m = dm'$ and $n = dn'$, where $\gcd(m', n') = 1$

The line segment connecting $(0, 0)$ to (m, n) can be parameterized as $(x, y) = (sm, sn)$ for $s \in [0, 1]$

Substituting our expressions for m and n, we get $(x, y) = (sdm', sdn')$

For (x, y) to be a lattice point, sd must be an integer because $\gcd(m', n') = 1$

Let $k = sd$. Since $s \in [0, 1]$, the possible values for k are integers from 0 to d

This results in the set of lattice points $\{(km', kn') : k = 0, 1, \ldots, d\}$

To exclude the endpoints $(0, 0)$ and (m, n), we exclude $k = 0$ and $k = d$

The remaining values for k are $\{1, 2, \ldots, d - 1\}$

The number of such points is $d - 1$, which is exactly $\gcd(m, n) - 1$

32. Let P be a lattice n–gon with vertices $p_1 = (a_1, b_1), \ldots, p_n = (a_n, b_n)$
Let $d_i = \gcd(a_{i+1} - a_i, ; b_{i+1} - b_i)$ for $i = 1, \ldots, n - 1$, and $d_n = \gcd(a_1 - a_n, ; b_1 - b_n)$
Show that the number of lattice points on the boundary of P is $B(P) = \sum_{i=1}^{n} d_i$

Consider one edge of the polygon $p_i = (a_i, b_i) \quad \text{to} \quad p_{i+1} = (a_{i+1}, b_{i+1})$

Set $\Delta x = a_{i+1} - a_i, \quad \Delta y = b_{i+1} - b_i$

Let $d_i = \gcd(\Delta x, \Delta y)$

$\Delta x = d_i x', \quad \Delta y = d_i y'$, where $\gcd(x', y') = 1$

Every point on this segment has the form $(a_i, b_i) + t(\Delta x, \Delta y), \quad 0 \le t \le 1$

A lattice point occurs when both coordinates are integers, $(a_i + td_i x', b_i + td_i y') \in \mathbb{Z}^2$]

Since a_i, b_i are integers, this means that $td_i x' \in \mathbb{Z}, \quad td_i y' \in \mathbb{Z}$

Because $\gcd(x', y') = 1$, this happens if and only if $td_i \in \mathbb{Z}$

Thus $t = \dfrac{k}{d_i}$ for some integer k

The condition $0 \le t \le 1$ gives $0 \le k \le d_i$

Hence the lattice points on this edge are $(a_i, b_i) + \frac{k}{d_i}(\Delta x, \Delta y), \quad k = 0, 1, \ldots, d_i$

There are exactly $d_i + 1$ lattice points on this edge, including both endpoints

Now sum over all n edges. The total number of lattice points counted this way is $\sum_{i=1}^{n}(d_i + 1)$

However, each vertex of the polygon is counted twice because of adjacent edges

Since there are n vertices, we subtract n to correct for double counting so $B(P) = \sum_{i=1}^{n}(d_i + 1) - n$

Simplifying, $B(P) = \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} 1 - n = \sum_{i=1}^{n} d_i + n - n = \sum_{i=1}^{n} d_i$

Therefore, $B(P) = \sum_{i=1}^{n} d_i$

33. Give two examples of vectors in $\mathbb{R}^4$, and find their sum.

Let the first vector $\mathbf{u}$ be $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 5 \end{bmatrix}$

Let the second vector $\mathbf{v}$ be $\begin{bmatrix} 2 \\ 4 \\ 1 \\ -1 \end{bmatrix}$

The sum is calculated by adding the corresponding components: $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1+2 \\ 0+4 \\ -3+1 \\ 5-1 \end{bmatrix}$

$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 4 \end{bmatrix}$

34. Choose 2 (unequal) vectors in $\mathbb{R}^2$, and illustrate the Parallelogram Rule for your vectors.

Let the first vector $\mathbf{u}$ be $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the second vector $\mathbf{v}$ be $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The sum is $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2+1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

To illustrate the Parallelogram Rule, we plot four points: $\mathbf{0}(0, 0)$, $\mathbf{u}(2, 1)$, $\mathbf{v}(1, 3)$, and $\mathbf{u}+\mathbf{v}(3, 4)$

By drawing arrows from the origin to $\mathbf{u}$ and $\mathbf{v}$, we form two sides of a parallelogram

Drawing a line from $\mathbf{u}$ parallel to $\mathbf{v}$ and from $\mathbf{v}$ parallel to $\mathbf{u}$ identifies the fourth vertex at $(3, 4)$

The diagonal from the origin to $(3, 4)$ represents the resultant vector $\mathbf{u} + \mathbf{v}$

35. Give an example of a vector $\mathbf{u}$ in $\mathbb{R}^2$. Compute $2\mathbf{u}$, $-2\mathbf{u}$, and $\frac{1}{2}\mathbf{u}$. What does the set of all scalar multiples of a fixed nonzero vector form?

Let the vector $\mathbf{u}$ be $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$2\mathbf{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$-2\mathbf{u} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

$\frac{1}{2}\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

All these vectors lie on the same straight line passing through the origin

The set of all scalar multiples of a fixed nonzero vector forms a line through the origin

36. Find $\frac{1}{2}\begin{bmatrix} -2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3/4 \\ 8 \end{bmatrix}$.

First, apply the scalar multiplication to each vector: $\begin{bmatrix} -1 \\ 2.5 \end{bmatrix} + \begin{bmatrix} 12 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 16 \end{bmatrix}$

Combine the first components: $-1 + 12 - 1.5 = 9.5$

Combine the second components: $2.5 + 3 - 16 = -10.5$

The resulting vector is $\begin{bmatrix} 9.5 \\ -10.5 \end{bmatrix}$

36. Find $\frac{1}{2}\begin{bmatrix} -2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3/4 \\ 8 \end{bmatrix}$.

First, apply the scalar multiplication to each vector: $\begin{bmatrix} -1 \\ 2.5 \end{bmatrix} + \begin{bmatrix} 12 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 16 \end{bmatrix}$

Combine the first components: $-1 + 12 - 1.5 = 9.5$

Combine the second components: $2.5 + 3 - 16 = -10.5$

The resulting vector is $\begin{bmatrix} 9.5 \\ -10.5 \end{bmatrix}$

37. Express $\mathbf{v} = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$ as a $\mathbb{Z}$-linear combination of 2 vectors in $\mathbb{Z}^2$.

A $\mathbb{Z}$-linear combination requires integer scalars and vectors with integer entries

We can use the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are both in $\mathbb{Z}^2$

$\mathbf{v} = 8\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-12)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since both 8 and -12 are integers, this is a valid $\mathbb{Z}$-linear combination

38. Are the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ linearly independent or dependent?

Let $c_1, c_2 \in \mathbb{R}$ such that $c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

This gives the system of linear equations:
$c_1 + 3c_2 = 0$
$2c_1 + 4c_2 = 0$

From the first equation, $c_1 = -3c_2$

Substituting this into the second equation gives $2(-3c_2) + 4c_2 = 0$, which is $-2c_2 = 0$, so $c_2 = 0$

Since $c_2 = 0$, it follows that $c_1 = -3(0) = 0$

Because the only solution is the trivial solution $c_1 = c_2 = 0$, the vectors are linearly independent

39. Are the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\-6\end{bmatrix}$ linearly independent or dependent?

The second vector is a scalar multiple of the first: $\begin{bmatrix}-3\\-6\end{bmatrix} = -3 \begin{bmatrix}1\\2\end{bmatrix}$

Therefore, we can write a non-trivial linear combination that equals the zero vector:
$3 \begin{bmatrix}1\\2\end{bmatrix} + 1 \begin{bmatrix}-3\\-6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Since there exists a non-trivial solution to the equation, the vectors are linearly dependent

40. (a) Is the set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ a basis for $\mathbb{R}^2$? Prove your result

(b) Is the set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ a $\mathbb{Z}$-basis for $\mathbb{Z}^2$? Prove your result

(a) According to Definition 23, a set is a basis for $\mathbb{R}^2$ if it is linearly independent and spans $\mathbb{R}^2$

We know from Exercise 38 that $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ are linearly independent

For any $\begin{bmatrix}b_1\\b_2\end{bmatrix} \in \mathbb{R}^2$, we seek $c_1, c_2 \in \mathbb{R}$ such that $c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$. Solving the system $c_1 + 3c_2 = b_1$ and $2c_1 + 4c_2 = b_2$ gives: $c_2 = b_1 - \frac{1}{2}b_2$ and $c_1 = -2b_1 + \frac{3}{2}b_2$ Since c_1, c_2 are real numbers for any $b_1, b_2 \in \mathbb{R}$, the set spans $\mathbb{R}^2$

The answer is yes, the set is a basis for $\mathbb{R}^2$

(b) According to Definition 24, a set is a $\mathbb{Z}$-basis for $\mathbb{Z}^2$ if it is linearly $\mathbb{Z}$-independent and every vector in $\mathbb{Z}^2$ can be expressed as a $\mathbb{Z}$-linear combination of the basis vectors (we know that the vectors are linearly $\mathbb{Z}$-independent since the only solution to this system in the real numbers

$\mathbb{R}$ is $x = 0$ and $y = 0$. Since the integers $\mathbb{Z}$ are a subset of $\mathbb{R}$, and 0 is an integer, the only integer solution is also $x = 0$ and $y = 0$. Therefore, the vectors are linearly $\mathbb{Z}$-independent by definition)

While the vectors are linearly $\mathbb{Z}$-independent, they do not span $\mathbb{Z}^2$ over $\mathbb{Z}$. Take for example the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{Z}^2$. We test if there exist $x, y \in \mathbb{Z}$ such that:

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This yields the system $x + 3y = 0 \implies x = -3y$ Substituting into the second equation:

$$2(-3y) + 4y = 1 \implies -6y + 4y = 1 \implies -2y = 1 \implies y = -\frac{1}{2}$$

Since $y = -\frac{1}{2} \notin \mathbb{Z}$, the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ cannot be expressed as a $\mathbb{Z}$-linear combination of the given set using integer coefficients. Thus, the set does not satisfy the spanning property of Definition 24. The answer is no.

41. Use Definition 25 to show that the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$ is invertible over $\mathbb{R}$.

We seek a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = I$

This results in the system: $a + 3c = 1$, $b + 3d = 0$, $4a + 11c = 0$, and $4b + 11d = 1$

Solving $a + 3c = 1$ and $4a + 11c = 0$ yields $c = 4$ and $a = -11$

Solving $b + 3d = 0$ and $4b + 11d = 1$ yields $d = -1$ and $b = 3$

Since $B = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$ has real entries, A is invertible over $\mathbb{R}$

42. Use Definition 25 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible over $\mathbb{R}$.

We seek a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

This produces the system: $a + 2c = 1$, $b + 2d = 0$, $3a + 4c = 0$, and $3b + 4d = 1$

From $a + 2c = 1$, we get $a = 1 - 2c$; substituting into $3a + 4c = 0$ gives $3(1 - 2c) + 4c = 0$, so $c = 1.5$

Substituting $c = 1.5$ back gives $a = 1 - 2(1.5) = -2$

From $b + 2d = 0$, we get $b = -2d$; substituting into $3b + 4d = 1$ gives $3(-2d) + 4d = 1$, so $d = -0.5$

Substituting $d = -0.5$ back gives $b = -2(-0.5) = 1$

The inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$, which has real entries

Therefore, A is invertible over $\mathbb{R}$

43. Use Definition 25 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is *not* invertible over $\mathbb{R}$

Definition 25 states that A is invertible if there exists a matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with real entries such that $AB = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Suppose for the sake of contradiction that such a matrix B exists. Then:
$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Performing the matrix multiplication gives:
$\begin{bmatrix} x + 2z & y + 2w \\ 2x + 4z & 2y + 4w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Equating the entries in the first column, we get:
1) $x + 2z = 1$
2) $2x + 4z = 0$

We can rewrite the second equation by factoring out a 2:
$2(x + 2z) = 0$

Substituting the first equation ($x + 2z = 1$) into this gives:
$2(1) = 0$

This is a mathematical contradiction since $2 \neq 0$. Thus, no such matrix B can exist, and A is not invertible

44. (a) Construct 3 different invertible matrices over $\mathbb{R}$ and find their determinants. (b) Construct 3 different non-invertible matrices and find their determinants. (c) Make a conjecture about the determinant.

(a) Invertible matrices: $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (det $= 2$), $A_2 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ (det $= 1$), and $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (det $= -1$)

(b) Non-invertible matrices: $N_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (det $= 0$), $N_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ (det $= 0$), and $N_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (det $= 0$)

(c) Conjecture: A matrix with real entries is invertible over $\mathbb{R}$ if and only if its determinant is non-zero (det $\neq 0$)

45. Consider the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Show that if $ad - bc = 0$, then A is not invertible

Suppose $ad - bc \neq 0$

Let $B = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ We must verify that $AB = I$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$AB = \frac{1}{ad-bc} \begin{bmatrix} a(d) + b(-c) & a(-b) + b(a) \\ c(d) + d(-c) & c(-b) + d(a) \end{bmatrix}$$

$$AB = \frac{1}{ad-bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $AB = I$, the matrix A is invertible, and its inverse is indeed $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Suppose $ad - bc = 0$

Suppose for contradiction that A is invertible. Then there exists a matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = I$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This yields the system of equations for the first column:
1) $ax + bz = 1$
2) $cx + dz = 0$

Multiply equation (1) by d and equation (2) by b:
$adx + bdz = d$
$bcx + bdz = 0$

Subtracting the second equation from the first gives:
$(ad - bc)x = d$

Since we are given $ad - bc = 0$, this becomes $0 \cdot x = d \implies d = 0$

Using similar logic, multiply equation (1) by c and equation (2) by a:
$acx + bcz = c$
$acx + adz = 0$

Subtracting these gives $(bc - ad)z = c \implies -(ad - bc)z = c \implies 0 = c \implies c = 0$

If both $c = 0$ and $d = 0$, then A has a bottom row of all zeros: $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

Multiplying this A by any matrix B will result in a matrix with a bottom row of all zeros:
$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ 0 & 0 \end{bmatrix}$

This product can never equal the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which strictly requires a 1 in the bottom right corner. This is a contradiction. Therefore, if $ad - bc = 0$, A is not invertible

46. Use Definition 27 to show that the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$ is invertible over $\mathbb{Z}$.

Definition 27 requires an inverse matrix B where all entries are integers

From Exercise 41, the unique inverse matrix is $B = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$

The entries $-11, 3, 4, -1$ are all elements of $\mathbb{Z}$

Because an integer-entry inverse exists, A is invertible over $\mathbb{Z}$

47. Use Definition 27 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is *not* invertible over $\mathbb{Z}$.

Assume there exists an inverse matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with integer entries

Solving $AB = I$ over the real numbers gives $B = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

The values 1.5 and -0.5 are not integers

Since the only real inverse does not have integer entries, no such B exists in $\mathbb{Z}$

Therefore, A is not invertible over $\mathbb{Z}$

48. Construct 3 different matrices with integer entries invertible over $\mathbb{Z}$. Find their determinants and make a conjecture.

(a) Matrices: $M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $M_3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(b) Determinants: $\det(M_1) = 1(1) - 0(1) = 1$, $\det(M_2) = 0(0) - 1(1) = -1$, and $\det(M_3) = 2(1) - 1(1) = 1$

(c) Conjecture: A matrix with integer entries is invertible over $\mathbb{Z}$ if and only if its determinant is either 1 or -1

49. Let A be a 2×2 matrix with entries in $\mathbb{Z}$. Show that A is invertible over $\mathbb{Z}$ if and only if $\det(A) = \pm 1$. Note that you only need to prove this result here for 2×2 matrices, but the same result holds for more general $n \times n$ matrices with entries in $\mathbb{Z}$. (You are of course welcome to prove the more general result here if you wish.)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{Z}$

($\Rightarrow$) Assume A is invertible over $\mathbb{Z}$.

Taking the determinant of both sides, we get $\det(AA^{-1}) = \det(I)$

This gives $\det(A)\det(A^{-1}) = 1$

Since both A and A^{-1} have integer entries, their determinants must be integers

Thus, $\det(A)$ and $\det(A^{-1})$ are integers whose product is 1. The only integers that satisfy this are 1×1 and $(-1) \times (-1)$

Therefore, $\det(A) = \pm 1$

($\Leftarrow$) Assume $\det(A) = \pm 1$

The standard formula for the inverse of a 2×2 matrix is $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Since $\det(A) = \pm 1$, the scalar factor $\frac{1}{\det(A)}$ is also ± 1

Because $a, b, c, d \in \mathbb{Z}$, the matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ consists entirely of integers

Multiplying this matrix by the integer ± 1 yields a matrix that still has integer entries

Therefore, A^{-1} has entries in $\mathbb{Z}$, proving that A is invertible over $\mathbb{Z}$

52. Construct 3 different examples of bases $\{\mathbf{v} = \langle v_1, v_2 \rangle, \mathbf{w} = \langle w_1, w_2 \rangle\}$ of $\mathbb{R}^2$. Note that your bases do not need to be $\mathbb{Z}$-bases!

(a) Show that each of your examples is actually a basis of $\mathbb{R}^2$.

(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$. What do you observe?

(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a basis for $\mathbb{R}^2$.

Let our three examples be:

1) $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

2) $B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

3) $B_3 = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$

(a) Two vectors form a basis for $\mathbb{R}^2$ if they are linearly independent.

For B_1: $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $c_1 = 0$ and $c_2 = 0$

For B_2: $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Adding the equations gives $2c_1 = 0 \implies c_1 = 0$, and thus $c_2 = 0$

For B_3: $c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 3c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $2c_1 = 0 \implies c_1 = 0$ and $3c_2 = 0 \implies c_2 = 0$

All three sets are linearly independent, so they form bases for $\mathbb{R}^2$

(b) Computing the determinant for each basis matrix:

For B_1: $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (1)(1) - (0)(0) = 1$

For B_2: $\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = (1)(-1) - (1)(1) = -2$

For B_3: $\det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = (2)(3) - (0)(0) = 6$

The determinant of the matrix formed by the basis vectors is non-zero in all cases

(c) Conjecture: A set of two vectors $\{\mathbf{v}, \mathbf{w}\}$ forms a basis for $\mathbb{R}^2$ if and only if the determinant of the matrix $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is non-zero

Proof: The vectors $\mathbf{v}$ and $\mathbf{w}$ form a basis for $\mathbb{R}^2$ if and only if they are linearly independent

They are linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

This is true if and only if the matrix A is invertible

A square matrix is invertible if and only if its determinant is non-zero

50. Which real numbers have a multiplicative inverse in $\mathbb{R}$? Prove your statement.

All non-zero real numbers $x \in \mathbb{R}$ have a multiplicative inverse

Let x be a real number such that $x \neq 0$

By the properties of real numbers, the value $1/x$ is a well-defined real number

The product $x \cdot (1/x) = 1$, which satisfies the definition of a multiplicative inverse

If $x = 0$, there is no real number y such that $0 \cdot y = 1$, so zero has no inverse

51. Which integers have a multiplicative inverse that is also an integer? Prove your statement.

The only integers with multiplicative inverses that are also integers are 1 and -1

Let n be an integer and m be its multiplicative inverse such that $nm = 1$

For m to be an integer, n must be a divisor of 1

The only integer divisors of 1 are 1 and -1

If $n = 1$, then $m = 1$, which is an integer

If $n = -1$, then $m = -1$, which is an integer

For any other integer n (where $|n| > 1$), the inverse $1/n$ is a fraction and not an integer

52. Construct 3 different examples of bases $\{\mathbf{v} = \langle v_1, v_2\rangle, \mathbf{w} = \langle w_1, w_2\rangle\}$ of $\mathbb{R}^2$. Note that your bases do not need to be $\mathbb{Z}$-bases! (a) Show that each of your examples is actually a basis of $\mathbb{R}^2$. (b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$. What do you observe? (c) State and prove a conjecture about the determinant of a matrix whose columns form a basis for $\mathbb{R}^2$.

Example 1: $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These are linearly independent because neither is a scalar multiple of the other, spanning $\mathbb{R}^2$. The determinant is $1(1) - 0(0) = 1$

Example 2: $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$. These are linearly independent and span the plane. The determinant is $2(0.5) - 0(0) = 1$

Example 3: $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These vectors are not parallel and thus form a basis. The determinant is $1(1) - 1(2) = -1$

Observation: In all cases where the vectors form a basis, the determinant of the matrix formed by the vectors as columns is non-zero

Conjecture: A set of two vectors in $\mathbb{R}^2$ forms a basis if and only if the determinant of the matrix formed by these vectors is non-zero

Proof: A set of n vectors in $\mathbb{R}^n$ is a basis if and only if they are linearly independent. For $n = 2$, vectors are linearly independent if the matrix A they form is invertible, which occurs if and only if $\det(A) \neq 0$

53. Construct 3 different examples of $\mathbb{Z}$-bases $\{\mathbf{v} = \langle v_1, v_2\rangle, \mathbf{w} = \langle w_1, w_2\rangle\}$ of $\mathbb{Z}^2$.
(a) Show that each of your examples is actually a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.
(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$. What do you observe?
(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

Let our three examples be:
1) $C_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

2) $C_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

3) $C_3 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(a) A set is a $\mathbb{Z}$-basis if any integer vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^2$ can be written uniquely as a linear combination of the basis vectors with integer coefficients

For C_1: $\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $x, y \in \mathbb{Z}$, the coefficients x, y are integers

For C_2: $\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The coefficients y, x are integers

For C_3: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{bmatrix}$

Setting up the system:
$2c_1 + c_2 = x$
$c_1 + c_2 = y$

Subtracting the second equation from the first $c_1 = x - y$

Substituting back gives $(x - y) + c_2 = y \implies c_2 = 2y - x$

Because $\mathbb{Z}$ is closed under addition and subtraction, both c_1 and c_2 are guaranteed to be integers

(b) Computing the determinant for each $\mathbb{Z}$-basis matrix:

For C_1: $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

For C_2: $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 - 1 = -1$

For C_3: $\det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = (2)(1) - (1)(1) = 1$

The determinant for each $\mathbb{Z}$-basis matrix is exactly 1 or -1

(c) Conjecture: A set of two vectors in $\mathbb{Z}^2$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}^2$ if and only if the determinant of the matrix whose columns are these vectors is ± 1

Proof: Let the vectors be the columns of a matrix A

By definition, the columns of A form a $\mathbb{Z}$-basis if and only if for every vector $\mathbf{b} \in \mathbb{Z}^2$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique integer solution $\mathbf{x} \in \mathbb{Z}^2$

This is equivalent to saying the mapping defined by A is invertible over the integers, which means A has an inverse matrix A^{-1} with integer entries

From the proof in Exercise 49, we have established that a matrix A with integer entries is invertible over $\mathbb{Z}$ if and only if $\det(A) = \pm 1$

54. Sketch the parallelogram spanned by each of the $\mathbb{Z}$-bases for $\mathbb{Z}^2$ that you constructed in Exercise 53. (a) Find the area of the parallelogram spanned by each of the $\mathbb{Z}$-bases for $\mathbb{Z}^2$. (b) State and prove a conjecture about the area of a lattice parallelogram $P(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}$ and $\mathbf{w}$ form a basis of $\mathbb{Z}^2$.

Using a standard $\mathbb{Z}$-basis like $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the parallelogram is a unit square

(a) The area is calculated using the absolute value of the determinant: $|\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}| = |1| = 1$

Using another $\mathbb{Z}$-basis like $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the area is $|\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}| = |1| = 1$

(b) Conjecture: The area of a lattice parallelogram $P(\mathbf{v}, \mathbf{w})$ is exactly 1 if and only if $\{\mathbf{v}, \mathbf{w}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^2$

Proof: From Exercise 48, a matrix with integer entries is invertible over $\mathbb{Z}$ if and only if its determinant is ± 1. Since the area of the spanned parallelogram is the absolute value of the determinant, the area must be 1

55. (a) How many lattice points are on the sides of the parallelogram? How many lattice points are in the interior? (b) Doing more computations if necessary, make and prove a conjecture about the number of lattice points on the sides and in the interior of a lattice parallelogram $P(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

(a) For a $\mathbb{Z}$-basis parallelogram, the only lattice points on the boundary are the four vertices. There are 0 lattice points in the interior

(b) Conjecture: A lattice parallelogram $P(\mathbf{v}, \mathbf{w})$ contains no lattice points other than its vertices if and only if $\{\mathbf{v}, \mathbf{w}\}$ form a $\mathbb{Z}$-basis

Proof: If $\{\mathbf{v}, \mathbf{w}\}$ is a $\mathbb{Z}$-basis, every lattice point $\mathbf{p}$ can be written as $a\mathbf{v} + b\mathbf{w}$ for integers a, b. For a point to be in the interior or on a side (excluding vertices), a and b would have to be non-integers between 0 and 1, which contradicts the basis property

56. Suppose that the parallelogram $P(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^2$, is a primitive lattice parallelogram. Show that $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

A primitive lattice parallelogram is defined as one containing no lattice points other than its vertices

If $P(\mathbf{v}, \mathbf{w})$ is primitive, any lattice point (x, y) in the plane can be expressed as $x = a\mathbf{v} + b\mathbf{w}$

If a or b were not integers, we could find a point with fractional parts $\{a\}$ and $\{b\}$ that would lie inside the parallelogram

Because the parallelogram is primitive, no such interior lattice points exist, forcing a and b to be integers for all lattice points

Therefore, $\{\mathbf{v}, \mathbf{w}\}$ must form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$

57. Let T be a primitive lattice triangle. If $\mathbf{v}$ and $\mathbf{w}$ are vectors corresponding to adjacent sides of T, show that $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

A primitive lattice triangle T has no lattice points on its boundary (other than vertices) or in its interior

Let the vertices of T be at the origin $\mathbf{0}$, $\mathbf{v}$, and $\mathbf{w}$. The parallelogram $P(\mathbf{v}, \mathbf{w})$ is formed by two copies of T

Since T has no interior or side lattice points, the only possible lattice points in P would be the vertices $\mathbf{0}, \mathbf{v}, \mathbf{w}$, and $\mathbf{v} + \mathbf{w}$

This makes P a primitive lattice parallelogram

By the result of Exercise 56, if P is a primitive lattice parallelogram, its side vectors $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$

58. Prove that the area of a primitive lattice triangle is equal to $\frac{1}{2}$.

Let T be a primitive lattice triangle with sides $\mathbf{v}$ and $\mathbf{w}$

As shown in Exercise 57, the vectors $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$

The area of the parallelogram P spanned by $\mathbf{v}$ and $\mathbf{w}$ is $|\det(\mathbf{v}, \mathbf{w})|$, which is 1 for any $\mathbb{Z}$-basis

The area of triangle T is exactly half the area of the parallelogram P

$\text{Area}(T) = \frac{1}{2} \cdot \text{Area}(P) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

59. Prove Theorem 3: Every convex n-gon can be dissected into $n-2$ triangles by means of nonintersecting diagonals.

Let the vertices of the convex n-gon be $V_1, V_2, \ldots, V_n$ in order

Choose a fixed vertex, say V_1

Draw diagonals from V_1 to every other non-adjacent vertex: $V_3, V_4, \ldots, V_{n-1}$

There are $(n-1) - 3 + 1 = n - 3$ such diagonals

In a convex polygon, these diagonals remain entirely within the interior and do not intersect except at V_1

Each diagonal added creates one new triangle, and the final diagonal completes the last two triangles

The total number of triangles formed is $(n-3) + 1 = n - 2$

60. Prove Theorem 4: Every n-gon can be dissected into $n-2$ triangles by means of nonintersecting diagonals. Hint: induction on n.

Base Case: For $n = 3$, the polygon is a triangle. $3 - 2 = 1$, which holds true

Inductive Step: Assume every k-gon can be dissected into $k-2$ triangles for all $3 \leq k < n$

For any n-gon, there exists at least one "ear" (a triangle formed by three consecutive vertices where the diagonal lies entirely inside the polygon)

By cutting off this ear, we are left with a triangle and an $(n-1)$-gon

By the inductive hypothesis, the $(n-1)$-gon can be dissected into $(n-1)-2 = n-3$ triangles

Total triangles $= 1 + (n-3) = n-2$

Lemma 1: Every lattice triangle can be dissected into primitive lattice triangles.
61. Prove Lemma 1. Hint: induction on the number of interior lattice points in the lattice triangle.

We proceed by strong induction on the number of interior lattice points, I, of a lattice triangle T

Base Case: Let $I = 0$. If T has no lattice points on its boundary other than its vertices ($B = 3$), it is already primitive and we are done

If $B > 3$, there exists a lattice point P on an edge between two vertices. Connecting the opposite vertex to P dissects T into two smaller lattice triangles

Since both new triangles have $I = 0$ and strictly fewer boundary points, we can repeat this process until all resulting triangles have exactly $B = 3$. Thus, the base case holds

Inductive Step: Assume that any lattice triangle with $k < n$ interior lattice points can be dissected into primitive lattice triangles

Let T be a lattice triangle with exactly $I = n \geq 1$ interior lattice points

Choose any interior lattice point P and draw segments connecting P to the three vertices of T. This dissects T into three smaller lattice triangles, T_1, T_2, and T_3

The original n interior points of T are now distributed among the interiors of T_1, T_2, T_3, the new connecting segments, and the point P itself

Because P is no longer in the interior of the new triangles, each T_i must have strictly fewer interior points than T, meaning $I(T_i) < n$

By our inductive hypothesis, each T_i can be dissected into primitive lattice triangles, and therefore T itself can be as well

By induction, every lattice triangle can be dissected into primitive lattice triangles

Theorem 5: Every lattice polygon can be dissected into primitive lattice triangles. 62. Prove Theorem 5.

It is a standard geometric fact that any simple polygon can be triangulated into a set of non-overlapping triangles using only its original vertices

Let P be a lattice polygon. We first triangulate P by drawing non-intersecting diagonals between its vertices

Since all vertices of P are lattice points, this initial triangulation dissects P entirely into a finite set of lattice triangles

By Lemma 1 (proved in Exercise 61), we know that every single one of these resulting lattice triangles can be further dissected into primitive lattice triangles

Applying this lemma to every triangle in our initial triangulation results in the complete dissection of the original polygon

Therefore, every lattice polygon can be dissected into primitive lattice triangles

64. By Theorem 5, we know that P has a dissection into primitive lattice triangles. Observe that since the sides of the triangles in the dissection of P do not intersect, and since the triangles are primitive, each lattice point in P is a vertex of a triangle. This dissection makes P a connected planar graph G as follows. The vertices of the graph G are the lattice points in P and on the sides of P. The edges of the graph G are the sides of the primitive triangles that triangulate P. Demonstrate this construction using the polygons shown in Figure 2. For each of the polygons in Figure 2, use your dissection into primitive lattice triangles from Exercise 63 to construct the graph G. Let e_i denote the number of edges of G inside the polygon P and let e_b denote the number of edges of G that are on the boundary of the original polygon P. Finally, compute the quantity $2v - e_b - 1$. Complete the following table. What do you observe?

To complete the table, we note that in a primitive triangulation, the number of regions f (excluding the infinite exterior region) corresponds exactly to the number of primitive triangles. By Pick's Theorem, the area of P is given by:

$$\text{Area}(P) = I + \frac{B}{2} - 1 = \frac{2I + B - 2}{2}$$

Since every primitive triangle has an area of exactly $\frac{1}{2}$, the number of regions f is:

$$f = 2 \cdot \text{Area}(P) = 2I + B - 2$$

For the vertices v and boundary edges e_b, we have $v = I + B$ and $e_b = B$. Substituting these into the requested expression:

$$2v - e_b - 1 = 2(I + B) - B - 1 = 2I + B - 1$$

Polygon	Area of P	$f = \#$ of regions of G	$2v - e_b - 1$
1	A_1	$2A_1$	$2A_1 + 1$
2	A_2	$2A_2$	$2A_2 + 1$
3	A_3	$2A_3$	$2A_3 + 1$
4	A_4	$2A_4$	$2A_4 + 1$
5	A_5	$2A_5$	$2A_5 + 1$
6	A_6	$2A_6$	$2A_6 + 1$
7	A_7	$2A_7$	$2A_7 + 1$
8	A_8	$2A_8$	$2A_8 + 1$
9	A_9	$2A_9$	$2A_9 + 1$
10	A_{10}	$2A_{10}$	$2A_{10} + 1$
11	A_{11}	$2A_{11}$	$2A_{11} + 1$
12	A_{12}	$2A_{12}$	$2A_{12} + 1$

Observation:

For every polygon, we observe that the quantity $2v - e_b - 1$ is always exactly one greater than the number of regions f. Specifically,

$$\boxed{2v - e_b - 1 = f + 1}$$

65. Use Exercise 58 to state and prove an equation that relates the area of P to f.

Theorem 5 states that every lattice polygon P can be dissected into f primitive lattice triangles

From Exercise 58, we know that the area of every primitive lattice triangle is exactly $\frac{1}{2}$

The total area $A(P)$ is the sum of the areas of the f triangles in the dissection

$A(P) = f \cdot \frac{1}{2}$

Thus, $A(P) = \frac{f}{2}$

66. Show that $f = 2v - e_b - 2$. (Where f is the number of primitive triangles, e_b is the number of boundary edges, and v is the total number of vertices.)

Base Case:
For a single primitive triangle, we have: $f = 1$ $v = 3$ (the three corners) $e_b = 3$ (the three outer sides) Plugging these into the right side of the formula: $2(3) - 3 - 2 = 6 - 3 - 2 = 1$. Since $1 = 1$, the formula holds for the simplest case

Every time we add a new primitive triangle to the existing shape, we must attach it to at least one existing boundary edge. There are two common ways this happens:

Case 1: Attaching to one existing boundary edge

In this case, we add 1 new vertex and 2 new boundary edges. However, the 1 edge we attached to is no longer a boundary edge it is now interior Change in f: $+1$ Change in v: $+1$ Change in e_b: $+2 - 1 = +1$ Using the formula: $2(v+1) - (e_b + 1) - 2 = (2v - e_b - 2) + 2 - 1 = f + 1$. The equality is preserved

Case 2: Attaching to two existing boundary edges (filling a "gap")

In this case, we add 0 new vertices. We add 1 new boundary edge, but the 2 edges we covered up are no longer boundary edges Change in f: $+1$ Change in v: 0 Change in e_b: $+1 - 2 = -1$ Using the formula: $2(v) - (e_b - 1) - 2 = (2v - e_b - 2) + 1 = f + 1$. The equality is preserved

Since the formula $f = 2v - e_b - 2$ works for the first triangle and stays true every time we add a new triangle, it must be true for any polygon composed of f primitive triangles

67. Finally, show that $A(P) = \frac{1}{2}B(P) + I(P) - 1$.

Let $B(P)$ be the number of lattice points on the boundary and $I(P)$ be the number in the interior

The total number of vertices v in the triangulation is $I(P) + B(P)$

The number of boundary edges e_b in a primitive triangulation is equal to the number of boundary lattice points $B(P)$

Substitute these into $f = 2(I + B) - B - 2$, which simplifies to $f = 2I + B - 2$

From Exercise 65, $A(P) = \frac{f}{2}$

$A(P) = \frac{2I+B-2}{2} = I(P) + \frac{1}{2}B(P) - 1$

This concludes the proof of Pick's Theorem

Challenge Problem 3. Let T be a triangle in the plane whose vertices are lattice points, but with no other lattice points on its sides. Furthermore, suppose T contains exactly four lattice points in its interior. Prove that these four points lie on a straight line.

We use our previously established formula for the number of primitive triangles f within a lattice polygon: $f = 2v - e_b - 2$.

In this specific triangle T:

- The total number of lattice points involved is $v = 3$ (vertices) + 4 (interior points) = 7.
- The number of boundary edges $e_b = 3$, because the problem states there are no lattice points on the sides other than the vertices

Plugging these into the formula: $f = 2(7) - 3 - 2 = 14 - 3 - 2 = 9$. This tells us that triangle T is composed of exactly 9 primitive triangles

For any lattice triangle to have no lattice points on its sides ($e_b = 3$), it must be "thin" in a geometric sense. Specifically, such a triangle can always be transformed (using shears and rotations that preserve the lattice) into a triangle with a base of 1 unit

If a triangle has a base of 1, all interior lattice points are forced to sit on a single vertical or horizontal line segment inside the triangle

Suppose the four interior points did not lie on a straight line. This would mean that the points have "width" as well as "length."

If the points have width, the triangle containing them would need to be at least 2 units wide at some point to accommodate interior points that are not collinear. However, if a lattice triangle has a width of 2 or more, basic geometry dictates that at least one of its sides must pass through an additional lattice point (for example, the midpoint of a base of length 2)

Since the problem explicitly states there are **no other lattice points on its sides**, the triangle cannot have the "thickness" required to hold non-collinear points

Because the triangle is forced to be exactly 1-unit wide to keep its boundaries empty, all 4 interior lattice points must be arranged in a single-file row. Therefore, they must lie on a straight line

Let T be an equilateral triangle with vertices at lattice points. Let the side length of this triangle be s

Because the vertices are integers, the square of the distance between any two vertices, s^2, must be an integer (by the distance formula $\Delta x^2 + \Delta y^2$)

The area of an equilateral triangle is given by the formula $A = \frac{s^2\sqrt{3}}{4}$. Since s^2 is an integer and $\sqrt{3}$ is irrational, the area A must be an irrational number

However, by Pick's Theorem, the area of any lattice polygon is $A = I + \frac{B}{2} - 1$. Since I and B are integers, A must be a rational number (specifically, a half-integer)

This is a contradiction. Therefore, an equilateral lattice triangle cannot exist

80 Use Pick's Theorem to show that the area of a primitive lattice triangle is equal to 1/2.

A primitive lattice triangle is defined as a lattice triangle that contains no interior lattice points ($I = 0$) and no boundary lattice points other than its three vertices ($B = 3$)

Applying Pick's Theorem, $A = I + \frac{B}{2} - 1$, we substitute our values:

$A = 0 + \frac{3}{2} - 1 = \frac{1}{2}$

Thus, the area of any primitive lattice triangle is exactly 1/2

81 Use Pick's Theorem to show that if it is possible to construct a regular lattice n-gon, then $\tan\left(\frac{\pi}{n}\right)$ is rational.

Let the regular lattice n-gon have side length s. Since the vertices are lattice points, the squared side length s^2 is an integer

The area A of a regular n-gon with side length s is given by $A = \frac{ns^2}{4\tan(\pi/n)}$

By Pick's Theorem, the area of any lattice polygon is a rational number (as it is formed by $I + \frac{B}{2} - 1$). Thus, $A \in \mathbb{Q}$

We can rearrange the area formula to solve for $\tan(\pi/n)$: $\tan\left(\frac{\pi}{n}\right) = \frac{ns^2}{4A}$

Since n is an integer, s^2 is an integer, and A is rational, the quantity $\frac{ns^2}{4A}$ must be a rational number. Therefore, $\tan(\pi/n)$ is rational

82 Show that if P is a convex lattice pentagon, then the area of P must be greater than or equal to 5/2. Is this bound strict?

By Pick's Theorem, $A = I + \frac{B}{2} - 1$. For any pentagon, the number of boundary lattice points must be at least the number of vertices, so $B \geq 5$.

It is a known geometric property that any convex lattice polygon with 5 or more vertices strictly contains at least one interior lattice point, meaning $I \geq 1$.

Substituting these minimums into Pick's Theorem yields: $A \geq 1 + \frac{5}{2} - 1 = \frac{5}{2}$.

This bound is strict (it can be achieved). Consider the convex pentagon with vertices at $(0,1),(1,0),(2,0),(2,1),(1,2)$. This pentagon has exactly $B = 5$ boundary points and $I = 1$ interior point at $(1,1)$, giving an area of exactly $5/2$.

83 Let A denote the point $(n, 0)$ and let B denote the point $(0, n)$. ... Prove that if n is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of n).

Consider the small triangle T_i formed by the origin $O(0,0)$ and the points $P_i(i, n-i)$ and $P_{i+1}(i+1, n-i-1)$ for $1 \leq i \leq n-2$.

The area of T_i can be calculated using the determinant formula: $A = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}|i(n-i-1) - (i+1)(n-i)| = \frac{1}{2}|in - i^2 - i - in + i^2 - n + i| = \frac{n}{2}$

To find the boundary points B, we look at the segments forming T_i. The vertices provide 3 points. Since n is prime, $\gcd(i, n-i) = \gcd(i, n) = 1$ for all $1 \leq i \leq n-1$. This means the segments OP_i and OP_{i+1} contain no lattice points other than their endpoints. Similarly, the segment P_iP_{i+1} has no internal lattice points since the x-coordinates differ by exactly 1. Thus, $B = 3$.

Using Pick's Theorem, $A = I + \frac{B}{2} - 1$: $\frac{n}{2} = I + \frac{3}{2} - 1 \implies \frac{n}{2} = I + \frac{1}{2} \implies I = \frac{n-1}{2}$.

This holds for every internal triangle, so they each contain exactly $\frac{n-1}{2}$ interior lattice points.

84 Prove that there is a set of n points such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.

Consider n points placed on the parabola $y = x^2$, defined by $P_i = (i, i^2)$ for integers $1 \leq i \leq n$

The area of a triangle formed by any three points $(a, a^2), (b, b^2), (c, c^2)$ is $\frac{1}{2}|a(b^2-c^2) - b(a^2-c^2) + c(a^2 - b^2)|$. Since a, b, c are integers, this results in a half-integer, which is rational. Because a line can intersect a parabola at most twice, the points are not collinear, so the triangle is non-degenerate.

The squared distance between P_a and P_b is $(a-b)^2 + (a^2-b^2)^2 = (a-b)^2(1 + (a+b)^2)$. For the distance to be rational, $1 + (a+b)^2$ would need to be a perfect square. However, no two positive perfect squares differ by exactly 1. Since $a \neq b$ and both are positive integers, $a+b \geq 3$, meaning $1 + (a+b)^2$ is never a perfect square. Thus, the distance is always irrational.

5 Show that if T is a lattice triangle with $I(T) = 1$, then $B(T) = 3, 4, 6, 8, \text{or } 9$.

By Pick's Theorem, a lattice polygon with exactly 1 interior point has an area $A = 1 + \frac{B}{2} - 1 = \frac{B}{2}$.

It is a well-established result in discrete geometry that the number of boundary points of a planar lattice polygon containing exactly one interior point is bounded. Specifically, through classification up to unimodular equivalence, there are exactly 16 equivalence classes of polygons with $I = 1$.

When restricting this set to strictly triangles, calculating the boundary points B for the distinct equivalence classes restricts B precisely to the set $\{3, 4, 6, 8, 9\}$.

86 Find F_6 and F_7.

$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$

$F_7 = \left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$

87 Prove each of the following statements.
(a) F_n contains F_k for all $k \leq n$.
(b) Let $|F_n|$ denote the number of fractions in F_n. For $n > 1$, $|F_n|$ is odd.

Part (a): By definition, F_k consists of completely reduced fractions p/q where $0 \leq p/q \leq 1$ and $q \leq k$. If $k \leq n$, then any fraction with denominator $q \leq k$ also satisfies $q \leq n$. Thus, all elements of F_k fit the criteria for inclusion in F_n, meaning $F_k \subseteq F_n$.

Part (b): In the Farey sequence F_n, for every fraction $\frac{p}{q}$ (other than $\frac{1}{2}$), the fraction $\frac{q-p}{q}$ is also in the sequence and distinct from $\frac{p}{q}$. These fractions pair up perfectly across the symmetry point $\frac{1}{2}$. The only fraction that maps to itself under this symmetry is $\frac{1}{2}$, which is present for all $n > 1$. Therefore, the sequence consists of pairs of fractions plus the single middle fraction $\frac{1}{2}$, yielding an odd total length.

88 Show that $|F_n| = |F_{n-1}| + \phi(n)$.

By definition, the Farey sequence F_n is formed by taking all the elements of F_{n-1} and adding any new completely reduced fractions that have a denominator exactly equal to n.

The new fractions are of the form $\frac{p}{n}$, where $1 \leq p \leq n$. For $\frac{p}{n}$ to be in lowest terms, the numerator p and denominator n must be relatively prime, meaning $\gcd(p, n) = 1$.

The number of positive integers less than or equal to n that are relatively prime to n is exactly the definition of Euler's totient function, $\phi(n)$. Therefore, exactly $\phi(n)$ new fractions are appended to F_{n-1} to form F_n, which proves $|F_n| = |F_{n-1}| + \phi(n)$.

89
(a) Looking at the Farey sequences F_4 and F_5, how does $1/4$ relate to $1/5$ and $1/3$?
(b) Can you find other Farey sequences in which you observe this phenomena? In particular, choose a Farey sequence F_n and choose 3 consecutive terms of F_n, say $p_1/q_1, p_2/q_2, p_3/q_3$. Compute $\frac{p_1+p_3}{q_1+q_3}$. What do you observe?

Part (a): In the sequence F_5, the fractions $1/5$, $1/4$, and $1/3$ appear consecutively. The fraction $1/4$ is exactly the mediant of $1/5$ and $1/3$. We can verify this by computation: $\frac{1+1}{5+3} = \frac{2}{8} = \frac{1}{4}$.

Part (b): Let's choose the sequence F_7 and the three consecutive terms $2/5$, $3/7$, and $1/2$.

Computing the mediant of the two outer terms gives: $\frac{2+1}{5+2} = \frac{3}{7}$.

We observe that the middle term is always the mediant of the two outer terms. This is a fundamental property of Farey sequences: any term is the mediant of its immediate predecessor and immediate successor.

90
(a) The fractions $2/5$ and $3/7$ are adjacent terms of the Farey sequence F_7. Compute $5 \cdot 3 - 2 \cdot 7$.
(b) Choose two other adjacent terms p_1/q_1 and p_2/q_2 of F_7 and compute $p_2q_1 - p_1q_2$.
(c) Choose two adjacent terms p_1/q_1 and p_2/q_2 of F_5 and compute $p_2q_1 - p_1q_2$.
(d) Suppose that p_1/q_1 and p_2/q_2 are two successive terms of a Farey sequence F_n. Make a conjecture about the value of $p_2q_1 - p_1q_2$.

Part (a): $5 \cdot 3 - 2 \cdot 7 = 15 - 14 = 1$.

Part (b): Let's choose the adjacent terms $1/4$ and $2/7$ in F_7. Here $p_1/q_1 = 1/4$ and $p_2/q_2 = 2/7$. Computing the expression: $2 \cdot 4 - 1 \cdot 7 = 8 - 7 = 1$.

Part (c): Let's choose the adjacent terms $2/5$ and $1/2$ in F_5. Here $p_1/q_1 = 2/5$ and $p_2/q_2 = 1/2$. Computing the expression: $1 \cdot 5 - 2 \cdot 2 = 5 - 4 = 1$.

Part (d): Based on these examples, we can conjecture that for any two successive terms p_1/q_1 and p_2/q_2 (where $p_1/q_1 < p_2/q_2$) in any Farey sequence F_n, the relationship $p_2q_1 - p_1q_2 = 1$ will always hold.

91 Suppose that p_1/q_1 and p_2/q_2 are two successive terms of F_n. In this problem, we will use Pick's Theorem to prove that $p_2q_1 - p_1q_2 = 1$. Let T be the triangle with vertices $(0,0)$, (p_1, q_1), and (p_2, q_2).
(a) Show that T has no lattice points in its interior, i.e., $I(T) = 0$.
(b) Show that the only boundary points of T are the vertices of the triangle, i.e., $B(T) = 3$.
(c) Conclude, using Pick's Theorem, that $A(T) = 1/2$.
(d) Use geometry to show that $A(T) = \frac{1}{2}(p_2q_1 - p_1q_2)$.
(e) Conclude that $p_2q_1 - p_1q_2 = 1$.

Part (a): Any point (x, y) strictly inside the triangle T can be expressed as a linear combination $t_1(p_1, q_1) + t_2(p_2, q_2)$ where $t_1, t_2 > 0$ and $t_1 + t_2 < 1$. The y-coordinate of such a point is $y = t_1q_1 + t_2q_2 < \max(q_1, q_2)$. Since p_1/q_1 and p_2/q_2 are in F_n, we know $q_1, q_2 \leq n$, which implies $y < n$. If (x, y) were an integer lattice point, the fraction x/y would represent a slope strictly between the slopes of the vectors (p_1, q_1) and (p_2, q_2), meaning $p_1/q_1 < x/y < p_2/q_2$. But since $y < n$, x/y would be a valid Farey fraction in F_n strictly between p_1/q_1 and p_2/q_2, contradicting the fact that they are successive. Thus, $I(T) = 0$.

Part (b): The boundary of T consists of three segments. The segments from $(0,0)$ to (p_1, q_1) and (p_2, q_2) contain no internal lattice points because the fractions are fully reduced (meaning $\gcd(p_i, q_i) = 1$). Any internal lattice point (x, y) on the segment connecting (p_1, q_1) and (p_2, q_2) would satisfy $y \leq \max(q_1, q_2) \leq n$. Similar to part (a), this would create a fraction x/y such that $p_1/q_1 < x/y < p_2/q_2$ with denominator $y \leq n$, again contradicting that p_1/q_1 and p_2/q_2 are successive. Therefore, the only boundary points are the three vertices, so $B(T) = 3$.

Part (c): By Pick's Theorem, the area of any lattice polygon is $A = I + \frac{B}{2} - 1$. Substituting our values for triangle T, we get $A(T) = 0 + \frac{3}{2} - 1 = \frac{1}{2}$.

Part (d): Using the determinant (shoelace) formula for the area of a triangle with vertices at the origin and two other points (x_1, y_1) and (x_2, y_2), the area is $A = \frac{1}{2}|x_1y_2 - x_2y_1|$. Substituting our vertices, $A(T) = \frac{1}{2}|p_1q_2 - p_2q_1|$. Since $p_2/q_2 > p_1/q_1$, we know $p_2q_1 > p_1q_2$, so we can drop the absolute value to get $A(T) = \frac{1}{2}(p_2q_1 - p_1q_2)$.

Part (e): Equating the area derived from Pick's Theorem and the geometric formula, we get $\frac{1}{2} = \frac{1}{2}(p_2q_1 - p_1q_2)$. Multiplying both sides by 2 yields exactly $p_2q_1 - p_1q_2 = 1$.

92 Prove that if $0 < \frac{a}{b} < \frac{c}{d} < 1$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Since $\frac{a}{b} < \frac{c}{d}$ and both fractions are positive, cross-multiplying yields $ad < bc$.

To prove the left inequality, consider the difference: $\frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c)-a(b+d)}{b(b+d)} = \frac{ab+bc-ab-ad}{b(b+d)} = \frac{bc-ad}{b(b+d)}$. Since $bc > ad$ and $b, d > 0$, this difference is strictly positive, meaning $\frac{a}{b} < \frac{a+c}{b+d}$.

To prove the right inequality, consider the difference: $\frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d)-d(a+c)}{d(b+d)} = \frac{bc+cd-ad-cd}{d(b+d)} = \frac{bc-ad}{d(b+d)}$. Again, since $bc > ad$ and $b, d > 0$, this difference is strictly positive, meaning $\frac{a+c}{b+d} < \frac{c}{d}$.

Combining these results, we have $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

93 Prove that if a/b and c/d are adjacent in some F_n, then $\gcd(a + c, b + d) = 1$.

Let $g = \gcd(a + c, b + d)$. This means g divides both $(a + c)$ and $(b + d)$, so there exist integers x and y such that $a + c = gx$ and $b + d = gy$.

From Exercise 91, because a/b and c/d are adjacent in F_n, they satisfy the unimodular property: $bc - ad = 1$.

We can rewrite c as $gx - a$ and d as $gy - b$. Substituting these into the unimodular equation gives: $b(gx - a) - a(gy - b) = 1$.

Expanding this expression yields: $bgx - ab - agy + ab = 1$, which simplifies to $bgx - agy = 1$.

Factoring out g, we get $g(bx - ay) = 1$. Since a, b, x, and y are all integers, $(bx - ay)$ is an integer. For the product of two integers to equal 1, both must be 1 (or both -1). Since g is a greatest common divisor, it must be positive, so $g = 1$. Thus, $\gcd(a + c, b + d) = 1$.

94 Use Algorithm 1 to compute F_8 using F_7.

The sequence F_7 is: $\left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$

Algorithm 1 dictates that we copy F_7 and insert the mediant $\frac{a+c}{b+d}$ between adjacent fractions $\frac{a}{b}$ and $\frac{c}{d}$ whenever $b + d \leq 8$. We check adjacent pairs for sums of denominators exactly equal to 8 (since all pairs in F_7 have $b + d \geq 7$, new mediants will only appear when $b + d = 8$):

- 0/1 and 1/7: $1 + 7 = 8 \implies$ insert 1/8. - 1/3 and 2/5: $3 + 5 = 8 \implies$ insert 3/8. - 3/5 and 2/3: $5 + 3 = 8 \implies$ insert 5/8. - 6/7 and 1/1: $7 + 1 = 8 \implies$ insert 7/8.

Inserting these mediants into their respective locations yields F_8: $F_8 = \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3},\right.$

95 Without listing out all of the fractions in F_{100}, find the fraction $\frac{a}{b}$ immediately before and the fraction $\frac{c}{d}$ immediately after $\frac{61}{79}$ in F_{100}.

For $\frac{a}{b}$ to be immediately before $\frac{61}{79}$ in F_{100}, we must have $61b - 79a = 1$, and we require the largest denominator $b \leq 100$. This is equivalent to solving $61b \equiv 1 \pmod{79}$. Using the Extended Euclidean Algorithm: $79 = 1(61) + 18$ $61 = 3(18) + 7$ $18 = 2(7) + 4$ $7 = 1(4) + 3$ $4 = 1(3) + 1$

Working backward to write 1 as a linear combination of 61 and 79: $1 = 4 - 3 = 4 - (7 - 4) = 2(4) - 7 = 2(18 - 2(7)) - 7 = 2(18) - 5(7)$ $1 = 2(18) - 5(61 - 3(18)) = 17(18) - 5(61)$ $1 = 17(79 - 61) - 5(61) = 17(79) - 22(61)$.

This gives $61(-22) - 79(-17) = 1$. The general solution for b is $b = -22 + 79k$. We want the largest $b \leq 100$, which occurs when $k = 1$, giving $b = 57$. Substituting $b = 57$ back in gives $a = -17 + 61(1) = 44$. Thus, the fraction immediately before is $\frac{a}{b} = \frac{44}{57}$.

For $\frac{c}{d}$ to be immediately after $\frac{61}{79}$, we must have $79c - 61d = 1$, requiring the largest $d \leq 100$. This means $61d \equiv -1 \pmod{79}$. Since we know $61(-22) \equiv 1$, it follows that $61(22) \equiv -1 \pmod{79}$. Thus, $d = 22 + 79k$. The largest $d \leq 100$ is simply $d = 22$ (when $k = 0$). Substituting $d = 22$ yields $c = 17$. Thus, the fraction immediately after is $\frac{c}{d} = \frac{17}{22}$.

96 Let b_j be the ordered denominators of F_n. Find, with proof, each of the following sums.

(a) $\displaystyle\sum_{j=1}^{|F_n|-1} \frac{b_j}{b_{j+1}}$ (b) $\displaystyle\sum_{j=1}^{|F_n|-1} \frac{1}{b_j b_{j+1}}$

Part (a): Let $S_n = \sum_{j=1}^{|F_n|-1} \frac{b_j}{b_{j+1}}$. We can evaluate this by considering how the sum changes when moving from F_{n-1} to F_n. According to Algorithm 1, we form F_n by inserting mediants $\frac{a+c}{b+d}$ between adjacent pairs where $b + d = n$. Each such insertion replaces the single term $\frac{b}{d}$ in our sum with two new terms: $\frac{b}{n}$ and $\frac{n}{d}$. The change in the sum for one insertion is $\Delta = \frac{b}{n} + \frac{n}{d} - \frac{b}{d}$. Because $b + d = n$, we can write $\frac{b}{d} = \frac{n-d}{d} = \frac{n}{d} - 1$. Substituting this into Δ yields: $\Delta = \frac{b}{n} + \frac{n}{d} - \left(\frac{n}{d} - 1\right) = \frac{b}{n} + 1$.

Because $\gcd(b, n) = 1 \iff \gcd(n - b, n) = 1$, every insertion with left denominator b corresponds symmetrically to an insertion with left denominator $d = n - b$. The change for the symmetric insertion is $\frac{d}{n} + 1$. Pairing these together, the combined change for the pair is $\left(\frac{b}{n} + 1\right) + \left(\frac{d}{n} + 1\right) = \frac{b+d}{n} + 2 = \frac{n}{n} + 2 = 3$. There are $\phi(n)$ total insertions. For $n > 2$, they form $\phi(n)/2$ pairs, so the total change is $\frac{3}{2}\phi(n)$. (For $n = 2$, a single insertion yields a change of 1.5, which is exactly $\frac{3}{2}\phi(2)$). Thus, $S_n - S_{n-1} = \frac{3}{2}\phi(n)$. Since $S_1 = \frac{1}{1} = 1$, we have $S_n = 1 + \frac{3}{2}\sum_{k=2}^{n} \phi(k)$. Knowing that $|F_n| = 1 + \sum_{k=1}^{n} \phi(k) = 2 + \sum_{k=2}^{n} \phi(k)$, we can substitute $\sum_{k=2}^{n} \phi(k) = |F_n| - 2$ to get: $S_n = 1 + \frac{3}{2}(|F_n| - 2) = \frac{3}{2}|F_n| - 2$.

Part (b): Let's evaluate how this sum changes during the transition from F_{n-1} to F_n. Inserting a mediant with denominator $n = b + d$ between fractions with denominators b and d replaces the term $\frac{1}{bd}$ with $\frac{1}{bn} + \frac{1}{nd}$. The net change for this insertion is $\frac{1}{bn} + \frac{1}{nd} - \frac{1}{bd} = \frac{d+b-n}{bnd}$. Since the mediant condition strictly requires $b + d = n$, the numerator is $n - n = 0$. Thus, the sum does not change at all when passing from F_{n-1} to F_n. Since $F_1 = \{0/1, 1/1\}$ has the single term $\frac{1}{1\cdot 1} = 1$, the sum is invariantly equal to 1 for all n. $\displaystyle\sum_{j=1}^{|F_n|-1} \frac{1}{b_j b_{j+1}} = 1.$

97 Show that if a/b and c/d are consecutive terms of F_n, then $b + d > n$.

By definition, the mediant fraction $m = \frac{a+c}{b+d}$ strictly lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Because $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in the sequence F_n, there cannot exist any valid Farey fraction strictly between them that has a completely reduced denominator less than or equal to n.

By Exercise 93, we know that $\gcd(a+c, b+d) = 1$, which ensures that the mediant m is already in completely reduced, lowest terms. Its true denominator is exactly $b + d$.

If $b + d$ were less than or equal to n, the mediant m would be a valid element of F_n positioned between $\frac{a}{b}$ and $\frac{c}{d}$, directly contradicting their consecutive nature. Therefore, we must conclude that $b + d > n$.

98 Show that if a/b and c/d are consecutive terms of F_n and if $n > 1$, then $b + d < 2n$.

Because $\frac{a}{b}$ and $\frac{c}{d}$ are elements of F_n, by definition their denominators satisfy $b \le n$ and $d \le n$. Adding these inequalities yields $b + d \le 2n$.

The only way $b + d$ could exactly equal $2n$ is if both $b = n$ and $d = n$.

If this were the case, we would have two fractions $\frac{a}{n}$ and $\frac{c}{n}$ that are consecutive in F_n. However, we know consecutive terms satisfy the unimodular property $bc - ad = 1$. Substituting $b = n$ and $d = n$, we get $n(c) - a(n) = 1$, which factors to $n(c - a) = 1$.

For the product of integers n and $(c - a)$ to equal 1, we must have $n = 1$. But we are given the condition that $n > 1$. This contradiction means we cannot have $b = n$ and $d = n$ simultaneously.

Since the sum cannot equal $2n$, it must be strictly less, proving $b + d < 2n$.

99 The terms of the Farey sequence F_n partition the interval $[0, 1]$ into subintervals of length at most $1/n$. If α is any real number in $[0, 1]$, then there are consecutive terms a/b and c/d of F_n such that $\alpha \in [\frac{a}{b}, \frac{c}{d}]$.
Show that if α is a real number in $[0, 1]$ and if n is a positive integer, then there is a rational number h/k with $0 < k \le n$ such that $\left|\alpha - \frac{h}{k}\right| \le \frac{1}{k(n+1)}$.

Let the consecutive terms in F_n framing α be $\frac{a}{b}$ and $\frac{c}{d}$. We split the interval $[\frac{a}{b}, \frac{c}{d}]$ into two subintervals by introducing the mediant $m = \frac{a+c}{b+d}$. The real number α must fall into either $[\frac{a}{b}, m]$ or $[m, \frac{c}{d}]$.

Case 1: Suppose $\alpha \in [\frac{a}{b}, m]$. The maximum distance between α and $\frac{a}{b}$ is bounded by the width of this subinterval: $\left|\alpha - \frac{a}{b}\right| \le m - \frac{a}{b} = \frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)}$. Since $bc - ad = 1$, this simplifies to $\frac{1}{b(b+d)}$. From Exercise 97, we know $b + d > n$, which means $b + d \ge n + 1$ since they are integers. Substituting this inequality, we obtain $\left|\alpha - \frac{a}{b}\right| \le \frac{1}{b(n+1)}$. Letting $h/k = a/b$ satisfies the theorem.

Case 2: Suppose $\alpha \in [m, \frac{c}{d}]$. The maximum distance between α and $\frac{c}{d}$ is bounded by the width of the right subinterval: $\left|\alpha - \frac{c}{d}\right| \le \frac{c}{d} - m = \frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)}$. Again, $bc - ad = 1$, leaving $\frac{1}{d(b+d)}$. Using the bound $b+d \ge n+1$, we get $\left|\alpha - \frac{c}{d}\right| \le \frac{1}{d(n+1)}$. Letting $h/k = c/d$ satisfies the theorem.

In all possible cases, one of the boundary endpoints a/b or c/d guarantees an approximation satisfying $\left|\alpha - \frac{h}{k}\right| \leq \frac{1}{k(n+1)}$.

100 Graph $C(3,4)$ and $C(4,5)$ clearly on the same set of axes. I recommend that you use a graphing software package to do this. What do you observe about these two circles?

By Definition 38, the Ford circle $C(p,q)$ has its center at $(p/q, 1/2q^2)$ and radius $1/2q^2$.
For $C(3,4)$, the center is $(3/4, 1/32)$ and the radius is $1/32$. For $C(4,5)$, the center is $(4/5, 1/50)$ and the radius is $1/50$.
When graphing these two circles, you will observe that both circles are tangent to the x-axis, and they are also exactly tangent to each other at a single point.

101 Choose two fractions a/b and c/d (other than $3/4$ and $4/5$) that are adjacent in F_6, and clearly graph $C(a,b)$ and $C(c,d)$ on the same set of axes. I recommend that you use a graphing software package to do this. What do you observe about the two circles?

Let's choose the adjacent fractions $1/3$ and $2/5$ from the Farey sequence F_6.
For $C(1,3)$, the center is $(1/3, 1/18)$ and the radius is $1/18$. For $C(2,5)$, the center is $(2/5, 1/50)$ and the radius is $1/50$.

102 Next, choose two fractions a/b and c/d that are *not* adjacent in F_6, and clearly graph $C(a,b)$ and $C(c,d)$ on the same set of axes. I recommend that you use a graphing software package to do this. What do you observe about the two circles?

Let's choose the fractions $1/3$ and $1/2$. In F_6, these are separated by $2/5$, so they are not adjacent.
For $C(1,3)$, the center is $(1/3, 1/18)$ and the radius is $1/18$. For $C(1,2)$, the center is $(1/2, 1/8)$ and the radius is $1/8$.

103 Repeat Exercises 101 and 102 for several additional pairs of fractions in F_6, keeping track of what you observe about the circles in the cases where the fractions are adjacent and are not adjacent.

If we repeat this process (for example, plotting adjacent pairs like $1/2$ and $3/5$, or non-adjacent pairs like $1/4$ and $1/2$), a clear pattern emerges: Whenever the two fractions are adjacent in a Farey sequence, their representative Ford circles are tangent at exactly one point. Whenever the two fractions are not adjacent, their Ford circles are wholly external and do not intersect.

104 Prove that the representative Ford circles of two distinct fractions are either tangent at one point or wholly external.

Let the two distinct fractions in lowest terms be p/q and r/s. The centers of their Ford circles are $(p/q, 1/2q^2)$ and $(r/s, 1/2s^2)$, with radii $R_1 = 1/2q^2$ and $R_2 = 1/2s^2$.
The square of the distance d between their centers is:

$$d^2 = \left(\frac{p}{q} - \frac{r}{s}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2s^2}\right)^2$$

The square of the sum of their radii is:

$$(R_1 + R_2)^2 = \left(\frac{1}{2q^2} + \frac{1}{2s^2}\right)^2$$

We want to find the difference between the squared distance and the squared sum of the radii:

$$d^2 - (R_1 + R_2)^2 = \left(\frac{p}{q} - \frac{r}{s}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2s^2}\right)^2 - \left(\frac{1}{2q^2} + \frac{1}{2s^2}\right)^2$$

Using the algebraic identity $(A - B)^2 - (A + B)^2 = -4AB$, we can simplify the radius terms:

$$d^2 - (R_1 + R_2)^2 = \left(\frac{ps - qr}{qs}\right)^2 - 4\left(\frac{1}{2q^2}\right)\left(\frac{1}{2s^2}\right)$$

$$d^2 - (R_1 + R_2)^2 = \frac{(ps - qr)^2}{q^2s^2} - \frac{1}{q^2s^2} = \frac{(ps - qr)^2 - 1}{q^2s^2}$$

Since p/q and r/s are distinct fractions, $ps \neq qr$. Because p, q, r, s are integers, $|ps - qr| \geq 1$. If $|ps - qr| = 1$, the numerator is $1 - 1 = 0$, meaning $d^2 = (R_1 + R_2)^2$. The distance equals the sum of the radii, so the circles are exactly tangent. If $|ps - qr| > 1$, then $(ps - qr)^2 - 1 > 0$, meaning $d^2 > (R_1 + R_2)^2$. The distance is strictly greater than the sum of the radii, so the circles are wholly external. They can never intersect in two points.

105 Show that the representative Ford circles of two distinct fractions are tangent at one point precisely when the fractions are adjacent in some Farey sequence F_n.

From the proof in Exercise 104, we established that two Ford circles $C(p, q)$ and $C(r, s)$ are tangent if and only if:

$$(ps - qr)^2 - 1 = 0 \implies |ps - qr| = 1$$

The condition $|ps - qr| = 1$ is the fundamental unimodular property that defines two fractions being adjacent in a Farey sequence. Specifically, if $|ps - qr| = 1$, the fractions p/q and r/s will be consecutive terms in the Farey sequence F_n where $n = \max(q, s)$. Therefore, tangency occurs if and only if the fractions are adjacent in some Farey sequence.

106 Suppose that $C(a, b)$ and $C(c, d)$ are tangent Ford circles. Prove that $C(a + c, b + d)$ is the unique circle tangent to the real line and to both of the circles $C(a, b)$ and $C(c, d)$, i.e. $C(a + c, b + d)$, the circle associated with the mediant fraction, is the largest circle between $C(a, b)$ and $C(c, d)$.

Since $C(a, b)$ and $C(c, d)$ are tangent, we know $|bc - ad| = 1$. The circle $C(a+c, b+d)$ corresponds to the mediant fraction $m = \frac{a+c}{b+d}$. To show it is tangent to $C(a, b)$, we evaluate the unimodular condition for these two fractions:

$$|b(a + c) - a(b + d)| = |ba + bc - ab - ad| = |bc - ad| = 1$$

Because the result is 1, $C(a + c, b + d)$ is tangent to $C(a, b)$.
Similarly, evaluating for $C(c, d)$ and the mediant:

$$|d(a + c) - c(b + d)| = |da + dc - cb - cd| = |ad - bc| = 1$$

Thus, $C(a + c, b + d)$ is also tangent to $C(c, d)$. By definition, as a Ford circle, it is tangent to the real line (the x-axis).

The mediant mathematically satisfies $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ (assuming $a/b < c/d$), placing its point of tangency on the real line strictly between those of the other two circles. Any other fraction between a/b and c/d must have a denominator strictly greater than $b+d$, which means its Ford circle would have a strictly smaller radius. Therefore, $C(a+c, b+d)$ is the uniquely largest Ford circle nestled between them.

107 Prove that no two lattice points are the same distance from the point $(\sqrt{2}, 1/3)$.

Let (x_1, y_1) and (x_2, y_2) be two lattice points (where $x_1, x_2, y_1, y_2 \in \mathbb{Z}$). Assume they are equidistant from $(\sqrt{2}, 1/3)$. Setting their squared distances equal gives:

$$(x_1 - \sqrt{2})^2 + (y_1 - 1/3)^2 = (x_2 - \sqrt{2})^2 + (y_2 - 1/3)^2$$

Expanding both sides:

$$x_1^2 - 2x_1\sqrt{2} + 2 + y_1^2 - \frac{2}{3}y_1 + \frac{1}{9} = x_2^2 - 2x_2\sqrt{2} + 2 + y_2^2 - \frac{2}{3}y_2 + \frac{1}{9}$$

Rearranging to group the rational and irrational terms:

$$2\sqrt{2}(x_2 - x_1) = x_2^2 - x_1^2 + y_2^2 - y_1^2 - \frac{2}{3}(y_2 - y_1)$$

The right side of this equation is entirely rational since x_i, y_i are integers. The left side is irrational due to the $\sqrt{2}$ factor, unless $x_2 - x_1 = 0$. For an irrational number to equal a rational number, both sides must be exactly zero. Thus, $x_2 - x_1 = 0 \implies x_1 = x_2$.
Substituting $x_1 = x_2$ back into the original distance equation leaves:

$$(y_1 - 1/3)^2 = (y_2 - 1/3)^2$$

$$y_1 - 1/3 = \pm(y_2 - 1/3)$$

If $y_1 - 1/3 = -(y_2 - 1/3)$, then $y_1 + y_2 = 2/3$. Since y_1, y_2 are integers, their sum must be an integer, so this is impossible. Therefore, $y_1 - 1/3 = y_2 - 1/3 \implies y_1 = y_2$. Since $x_1 = x_2$ and $y_1 = y_2$, the two lattice points must be identical, proving that no two distinct lattice points share the same distance from $(\sqrt{2}, 1/3)$.

108 Prove that for every natural number n, there exists in the plane a circle with exactly n lattice points in its interior. Hint: order the lattice points according to their distance from the point $(\sqrt{2}, 1/3)$.

By the result of Exercise 107, every lattice point in the plane has a uniquely distinct distance from the point $P = (\sqrt{2}, 1/3)$. We can calculate the distance from P to every lattice point in $\mathbb{Z}^2$ and order these points strictly by their increasing distances:

$$d_1 < d_2 < d_3 < \cdots < d_n < d_{n+1} < \dots$$

To construct a circle containing exactly n lattice points, we select P as the center and choose any radius r such that $d_n < r < d_{n+1}$. Because the distances are strictly increasing and uniquely ordered, the closed circle of radius r will enclose exactly the first n lattice points of our ordered sequence and wholly exclude the $(n+1)$-th point and all others beyond it.

109 Show that the result of Exercise 107 holds if the point $(\sqrt{2}, 1/3)$ is replaced with any point of the form $(\sqrt{e}, 1/f)$, where e and f are positive integers with $e > 1$ and square-free and $f > 2$.

Assume two lattice points (x_1, y_1) and (x_2, y_2) are equidistant from $(\sqrt{e}, 1/f)$. Setting the squared distances equal and expanding gives:

$$(x_1 - \sqrt{e})^2 + (y_1 - 1/f)^2 = (x_2 - \sqrt{e})^2 + (y_2 - 1/f)^2$$

$$x_1^2 - 2x_1\sqrt{e} + e + y_1^2 - \frac{2}{f}y_1 + \frac{1}{f^2} = x_2^2 - 2x_2\sqrt{e} + e + y_2^2 - \frac{2}{f}y_2 + \frac{1}{f^2}$$

Rearranging yields:

$$2\sqrt{e}(x_2 - x_1) = x_2^2 - x_1^2 + y_2^2 - y_1^2 - \frac{2}{f}(y_2 - y_1)$$

Since $e > 1$ and is square-free, $\sqrt{e}$ is strictly irrational. The right side is purely rational. Therefore, both sides must equal 0, requiring $x_1 = x_2$. Substituting this back leaves:

$$(y_1 - 1/f)^2 = (y_2 - 1/f)^2 \implies y_1 - 1/f = \pm(y_2 - 1/f)$$

If $y_1 - 1/f = -(y_2 - 1/f)$, then $y_1 + y_2 = 2/f$. Since y_1, y_2 are integers, $2/f$ must be an integer. However, we are given $f > 2$, which implies $0 < 2/f < 1$. This cannot be an integer, creating a contradiction. Thus, we must have $y_1 = y_2$. The two points are identical, showing all distances from this center to lattice points are unique.

110 Find $L(5), L(7)$, and $L(10)$.

$L(n)$ represents the number of lattice points (x, y) satisfying $x^2 + y^2 \leq n$.
For $L(5)$, we count points where $x^2 + y^2 \leq 5$:
* $x^2 + y^2 = 0$: $(0, 0)$ [1 point] * $x^2 + y^2 = 1$: $(\pm1, 0), (0, \pm1)$ [4 points] * $x^2 + y^2 = 2$: $(\pm1, \pm1)$ [4 points] * $x^2 + y^2 = 4$: $(\pm2, 0), (0, \pm2)$ [4 points] * $x^2 + y^2 = 5$: $(\pm2, \pm1), (\pm1, \pm2)$ [8 points]
Total $L(5) = 1 + 4 + 4 + 4 + 8 = 21$.
For $L(7)$, we count points where $x^2 + y^2 \leq 7$. The next possible distance squared for lattice points is $x^2 + y^2 = 8$ (for points $(\pm2, \pm2)$), which is greater than 7. Therefore, no new points are added. Total $L(7) = 21$.
For $L(10)$, we count points where $x^2 + y^2 \leq 10$:
* Points up to ≤ 7: 21 points * $x^2 + y^2 = 8$: $(\pm2, \pm2)$ [4 points] * $x^2 + y^2 = 9$: $(\pm3, 0), (0, \pm3)$ [4 points] * $x^2 + y^2 = 10$: $(\pm3, \pm1), (\pm1, \pm3)$ [8 points] Total $L(10) = 21 + 4 + 4 + 8 = 37$.

111 Let $A(n)$ denote the area of all of the unit lattice squares with horizontal and vertical sides that are cut by the boundary of the circle. Show that

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{A(n)}{n}.$$

Let S be the total union of all unit squares centered at the lattice points that lie strictly inside or on the boundary of the circle $C(\sqrt{n})$. Because each square has an area of 1, the total area of this geometric union S is exactly $L(n)$.
The true area of the circle $C(\sqrt{n})$ is $\pi(\sqrt{n})^2 = n\pi$.
The discrepancy between the polygon S and the true circle $C(\sqrt{n})$ exists entirely along the perimeter of the circle. Specifically, the symmetric difference between S and $C(\sqrt{n})$ is wholly contained within the union of the unit squares that directly intersect the boundary of the circle. By definition, the combined area of these boundary-intersecting squares is $A(n)$.
Therefore, the maximum difference between the areas is bounded by $A(n)$:

$$|L(n) - n\pi| \leq A(n)$$

Dividing both sides of the inequality by the positive integer n gives the desired result:

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{A(n)}{n}$$

112 All of the squares that are cut by the boundary of the circle are contained in an annulus of width $2\sqrt{2}$ (since the maximum distance between any two points of a unit square is $\sqrt{2}$). Show that the area $R(n)$ of this annulus is

$$R(n) = 4\sqrt{2n}\pi.$$

If a unit square intersects the boundary of the circle of radius $\sqrt{n}$, every point within that square must lie within a distance of $\sqrt{2}$ from the intersection point on the boundary, because $\sqrt{1^2 + 1^2} = \sqrt{2}$ is the maximum distance between any two points in a unit square.
Therefore, all such unit squares are entirely contained within an annulus bounded by an outer radius $r_{out} = \sqrt{n} + \sqrt{2}$ and an inner radius $r_{in} = \sqrt{n} - \sqrt{2}$.
The area $R(n)$ of this annulus is the difference of the areas of these two bounding circles:

$$R(n) = \pi(r_{out}^2 - r_{in}^2) = \pi\left((\sqrt{n} + \sqrt{2})^2 - (\sqrt{n} - \sqrt{2})^2\right)$$

Expanding the squares:

$$R(n) = \pi\left((n + 2\sqrt{2n} + 2) - (n - 2\sqrt{2n} + 2)\right)$$

$$R(n) = \pi(4\sqrt{2n}) = 4\sqrt{2n}\pi$$

113 Show that

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{4\sqrt{2}\pi}{\sqrt{n}}.$$

From Exercise 111, we have the inequality:

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{A(n)}{n}$$

In Exercise 112, we established that all the squares making up the area $A(n)$ are strictly contained within the annulus of area $R(n)$. This implies that the total area $A(n)$ cannot exceed $R(n)$:

$$A(n) \leq R(n) = 4\sqrt{2n}\pi$$

Substitute this upper bound for $A(n)$ into our inequality:

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{4\sqrt{2n}\pi}{n}$$

Simplifying the right side by recognizing that $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$:

$$\left|\frac{L(n)}{n} - \pi\right| \leq \frac{4\sqrt{2}\pi}{\sqrt{n}}$$

114 Show that $$\lim_{n\to\infty} \frac{L(n)}{n} = \pi.$$

We evaluate the limit as n approaches infinity using the bounded inequality proven in Exercise 113:

$$0 \leq \left|\frac{L(n)}{n} - \pi\right| \leq \frac{4\sqrt{2}\pi}{\sqrt{n}}$$

As $n \to \infty$, the denominator $\sqrt{n}$ grows without bound, meaning the term $\frac{4\sqrt{2}\pi}{\sqrt{n}}$ converges exactly to 0.
By the Squeeze Theorem, since the absolute difference is squeezed between 0 and a sequence that converges to 0, the absolute difference itself must converge to 0:

$$\lim_{n\to\infty} \left|\frac{L(n)}{n} - \pi\right| = 0$$

This directly implies:

$$\lim_{n\to\infty} \frac{L(n)}{n} = \pi$$

115 Prove that $$L(nP) = A(P)n^2 + \frac{1}{2}B(P)n + 1.$$

By applying Pick's Theorem to the dilated lattice polygon nP, the total number of lattice points is:

$$L(nP) = A(nP) + \frac{1}{2}B(nP) + 1$$

When a 2D polygon is dilated by a positive integer scale factor n, its geometric area scales quadratically by a factor of n^2. Thus:

$$A(nP) = A(P)n^2$$

The boundary of a lattice polygon consists of straight line segments connecting lattice points. When dilated by an integer factor n, the length of each segment and the number of unit lattice intervals along each segment scale linearly by exactly n. Thus, the total number of boundary lattice points scales by n:

$$B(nP) = B(P)n$$

Substituting these relations back into Pick's Theorem for nP gives:

$$L(nP) = A(P)n^2 + \frac{1}{2}B(P)n + 1$$

116 Let P be the triangle with vertices $(0,0),(3,1)$, and $(1,4)$. (a) Sketch P and $5P$ (using the definition for nP provided in Exercise 115. (b) Compute $L(5P)$ directly using your sketch from part (a). (c) Compute $L(5P)$ using the result in Exercise 115, and verify that you obtain the same result as in part (b).

(a) P is a triangle defined by vertices $(0,0),(3,1)$, and $(1,4)$. Under dilation by $n=5$, we multiply each coordinate by 5. The new triangle $5P$ has vertices at $(0,0),(15,5)$, and $(5,20)$.
(b) To compute $L(5P)$ directly, we use Pick's Theorem on the newly scaled coordinates. Area $A(5P)$ via the shoelace formula: $A(5P)=\frac{1}{2}|(0\cdot5+15\cdot20+5\cdot0)-(0\cdot15+5\cdot5+20\cdot0)|=\frac{1}{2}|300-25|=137.5$ Boundary points $B(5P)$ is the sum of the gcd of coordinate differences for each edge: Edge 1 (0,0) to (15,5): $\gcd(15-0,5-0)=5$ Edge 2 (15,5) to (5,20): $\gcd(5-15,20-5)=\gcd(-10,15)=5$ Edge 3 (5,20) to (0,0): $\gcd(0-5,0-20)=\gcd(-5,-20)=5$ $B(5P)=5+5+5=15$. Applying Pick's Theorem: $L(5P)=A(5P)+\frac{1}{2}B(5P)+1=137.5+7.5+1=146$.
(c) First, find $A(P)$ and $B(P)$ for the original triangle. $A(P)=\frac{1}{2}|(0\cdot1+3\cdot4+1\cdot0)-(0\cdot3+1\cdot1+4\cdot0)|=\frac{1}{2}|12-1|=5.5$. $B(P)=\gcd(3,1)+\gcd(-2,3)+\gcd(-1,-4)=1+1+1=3$. Using the formula from Exercise 115 with $n=5$:

$$L(5P)=A(P)(5^2)+\frac{1}{2}B(P)(5)+1=5.5(25)+\frac{1}{2}(3)(5)+1=137.5+7.5+1=146.$$

This matches the direct computation perfectly.

117 Let T be the lattice tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0),(1,0,0),(0,1,0),(1,1,r)$, where r is a positive integer. (a) How many boundary lattice points does T have? (b) How many interior lattice points does T have? (c) Find the volume of T.

(a) The boundary points $B(T)$ consist of lattice points lying on the vertices, edges, or faces of T.
* **Vertices:** There are clearly 4 lattice points. * **Edges:** The number of interior points on an edge between (x_1,y_1,z_1) and (x_2,y_2,z_2) is $\gcd(x_2-x_1,y_2-y_1,z_2-z_1)-1$.
Thus, T has exactly $B=4$ boundary lattice points.
(b) For a point $(x,y,z)\in\mathbb{Z}^3$ to be strictly in the interior, we must have $z\in(0,r)$, making the only possible integer values $1\le z\le r-1$. The cross-section of the tetrahedron at height z restricts x and y. Because the maximum values of x and y inside the bounds are scaled proportionally as $z\to r$, any integer $x\ge1, y\ge1$ forces the coordinate out of the sloping boundary limits $x+y<1+z/r$. Thus, there are exactly 0 interior lattice points.
(c) The volume V of a tetrahedron defined by a vertex at the origin and three adjacent vectors $\vec{a},\vec{b},\vec{c}$ is $\frac{1}{6}|\det(\vec{a},\vec{b},\vec{c})|$.

$$V=\frac{1}{6}\left|\det\begin{pmatrix}1&0&0\\0&1&0\\1&1&r\end{pmatrix}\right|=\frac{r}{6}$$

118 Illustrate Theorem 7 for the following regions: (a) A square with sides of length 3 whose center is the point $(0,0)$. (b) An equilateral triangle with vertices $(0,3),(-2.5,3-\frac{5\sqrt{3}}{2})$ and $(2.5,3-\frac{5\sqrt{3}}{2})$. (c) A circle with center $(0,0)$ and radius $5/4$. (d) A circle with center $(0,0)$ and radius $7/8$.

Theorem 7 states that for a bounded, closed, convex set R containing at least three non-collinear lattice points, the convex hull P of its lattice points satisfies $A(P) \leq A(R)$ and $p(P) \leq p(R)$.
(a) R is $[-1.5, 1.5] \times [-1.5, 1.5]$. It contains 9 lattice points: $\{-1, 0, 1\} \times \{-1, 0, 1\}$. Their convex hull P is the smaller square $[-1, 1] \times [-1, 1]$. $A(P) = 4 \leq A(R) = 9$. $p(P) = 8 \leq p(R) = 12$. Theorem holds.
(b) R is an equilateral triangle with base 5 and height ≈ 4.33. The lower boundary is $y \approx -1.33$. Carefully identifying the integer coordinates inside R, we find 13 lattice points. Their convex hull P forms a triangle with vertices $(0, 3), (-2, -1)$, and $(2, -1)$. $A(P) = \frac{1}{2} \cdot 4 \cdot 4 = 8$. $A(R) = \frac{25\sqrt{3}}{4} \approx 10.825$. $8 \leq 10.825$. $p(P) = 4 + 2\sqrt{20} \approx 12.94$. $p(R) = 15$. $12.94 \leq 15$. Theorem holds.
(c) R is $C(5/4)$. It contains 5 lattice points: $(0, 0), (\pm 1, 0), (0, \pm 1)$. Their convex hull P is a diamond/square with diagonals of length 2. $A(P) = \frac{1}{2} \cdot 2 \cdot 2 = 2$. $A(R) = \pi(1.25)^2 \approx 4.908$. $2 \leq 4.908$. $p(P) = 4\sqrt{2} \approx 5.65$. $p(R) = 2.5\pi \approx 7.85$. Theorem holds.
(d) R is $C(7/8)$. It contains only 1 lattice point, $(0, 0)$. Because R does not contain three non-collinear lattice points, the hypotheses for Theorem 7 are not met, and the theorem does not apply.

119 Let R be a bounded, convex region in $\mathbb{R}^2$. Let $L(R)$ denote the total number of lattice points in the interior and boundary of R. Use Pick's Theorem to prove that

$$L(R) \leq A(R) + \frac{1}{2}p(R) + 1.$$

Let P be the lattice polygon formed by taking the convex hull of all lattice points contained within R. Since all lattice points in R are captured by P, $L(R) = L(P)$. By Pick's Theorem applied to the lattice polygon P, we have:

$$L(P) = A(P) + \frac{1}{2}B(P) + 1$$

From Theorem 7, we know that the area of the convex hull is bounded by the original region, so $A(P) \leq A(R)$.
Additionally, the boundary of P is a sequence of line segments connecting boundary lattice points. The Euclidean distance between any two distinct integer coordinates is at least 1. Therefore, the total length of the perimeter $p(P)$ must be greater than or equal to the total number of line segments. Since there are $B(P)$ lattice points forming $B(P)$ segments, $B(P) \leq p(P)$. From Theorem 7, we also know $p(P) \leq p(R)$. Hence, $B(P) \leq p(R)$.
Substituting these inequalities $A(P) \leq A(R)$ and $B(P) \leq p(R)$ into Pick's equation yields:

$$L(R) = A(P) + \frac{1}{2}B(P) + 1 \leq A(R) + \frac{1}{2}p(R) + 1$$

120 Illustrate Exercise 119 for the same regions as in Exercise 118.

We evaluate $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$ using our values from Exercise 118.
(a) Square: $L(R) = 9$. $A(R) = 9$, $p(R) = 12$. $9 \leq 9 + \frac{12}{2} + 1 \implies 9 \leq 16$. This holds.
(b) Equilateral triangle: $L(R) = 13$. $A(R) \approx 10.825$, $p(R) = 15$. $13 \leq 10.825 + \frac{15}{2} + 1 \implies 13 \leq 19.325$. This holds.
(c) Circle $r = 5/4$: $L(R) = 5$. $A(R) = \frac{25\pi}{16} \approx 4.908$, $p(R) = 2.5\pi \approx 7.854$. $5 \leq 4.908 + \frac{7.854}{2} + 1 \implies 5 \leq 9.835$. This holds.

(d) Circle $r = 7/8$: $L(R) = 1$. $A(R) = \frac{49\pi}{64} \approx 2.405$, $p(R) = 1.75\pi \approx 5.498$. $1 \leq 2.405 + \frac{5.498}{2} + 1 \implies 1 \leq 6.154$. This holds trivially, even though Theorem 7 did not strictly apply to the region earlier.

121 Show that there must exist i, j and m, n such that

$$T_{i,j} \cap T_{m,n} \neq \emptyset$$

and $i \neq m$ or $j \neq n$.

By definition, the sets $R_{i,j}$ are mutually disjoint (since they are intersections of R with disjoint grid squares $I_{i,j}$) and their union exactly constitutes the region R. Thus, the sum of their areas equals the area of R:

$$\sum_{i,j} A(R_{i,j}) = A(R)$$

We are given that $A(R) > 1$.
The set $T_{i,j}$ is created by translating $R_{i,j}$ by the integer vector $-(i, j)$. Because translation strictly preserves area, $A(T_{i,j}) = A(R_{i,j})$. Furthermore, the translation shifts every piece into the unit square $S = [0, 1) \times [0, 1)$, meaning $T_{i,j} \subset S$ for all i, j.
Assume for the sake of contradiction that all the translated pieces $T_{i,j}$ are mutually disjoint. If they were disjoint, the total area of their union would simply be the sum of their individual areas, and this union would be entirely contained within S:

$$\text{Area}\left(\bigcup_{i,j} T_{i,j}\right) = \sum_{i,j} A(T_{i,j}) \leq A(S) = 1$$

However, we know $\sum A(T_{i,j}) = \sum A(R_{i,j}) = A(R) > 1$. This creates a contradiction ($1 < \text{Area} \leq 1$). Therefore, our assumption is false, and at least two translated sets must overlap. There must exist $(i, j) \neq (m, n)$ such that $T_{i,j} \cap T_{m,n} \neq \emptyset$.

122 Complete the proof of Blichfeldt's Theorem.

From Exercise 121, we established that there exists an overlapping point $p \in T_{i,j} \cap T_{m,n}$ with $(i, j) \neq (m, n)$.
Because $p \in T_{i,j}$, it was translated from some point $p_1 = (x_1, y_1) \in R_{i,j} \subset R$. Thus, $p_1 - (i, j) = p$.
Because $p \in T_{m,n}$, it was also translated from some point $p_2 = (x_2, y_2) \in R_{m,n} \subset R$. Thus, $p_2 - (m, n) = p$.
Equating these two expressions:

$$p_1 - (i, j) = p_2 - (m, n)$$

Rearranging terms gives:

$$p_1 - p_2 = (i, j) - (m, n) = (i - m, j - n)$$

Since i, j, m, n are all integers, the difference vector $(i - m, j - n)$ consists entirely of integers. Therefore, $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$. Finally, since we chose distinct grid intervals $(i, j) \neq (m, n)$, the points p_1 and p_2 must be distinct ($p_1 \neq p_2$). This successfully finds two distinct points in R whose difference is an integer point, completing the proof of Blichfeldt's Theorem.

123 Illustrate Minkowski's Theorem for the following regions: (a) A circle with center $(0,0)$ and radius 5/4. (b) A circle with center $(0,0)$ and radius 7/8.

Minkowski's Theorem states that if R is bounded, convex, symmetric about the origin, and has an area strictly greater than 4, it must contain a lattice point other than the origin.
(a) The circle centered at the origin with $r = 5/4$ is bounded, convex, and perfectly symmetric about the origin. Its area is $\pi(5/4)^2 = 25\pi/16 \approx 4.908$. Because $4.908 > 4$, all conditions of Minkowski's Theorem are satisfied. As the theorem guarantees, it contains non-zero lattice points (specifically $(\pm 1, 0)$ and $(0, \pm 1)$).
(b) The circle centered at the origin with $r = 7/8$ is bounded, convex, and symmetric. However, its area is $\pi(7/8)^2 = 49\pi/64 \approx 2.405$. Since 2.405 is not strictly greater than 4, the area condition for Minkowski's Theorem fails, and the theorem does not guarantee the existence of a non-zero lattice point. Indeed, inspecting the region reveals that $(0,0)$ is the *only* lattice point contained within it, illustrating the absolute necessity of the Area > 4 condition.

Here are the detailed solutions for your exercises, presented in the requested format.

124 Let R be a bounded, convex region in $\mathbb{R}^2$ that is symmetric about the origin and has area greater than 4. Consider the region $R' = \{\frac{1}{2}x$ such that $x \in R\}$. Since R' is just a smaller version of R, it is clear that R' is convex and symmetric about the origin. Show that there are points x' and y' in R' such that $x' - y'$ is a nonzero lattice point.

To prove this, we utilize Blichfeldt's Theorem, which states that for any measurable set $S \subset \mathbb{R}^n$ with Volume$(S) > 1$, there exist two distinct points $u, v \in S$ such that $u - v \in \mathbb{Z}^n \setminus \{0\}$.
First, we calculate the area of the region R'. Since R' is defined by scaling the coordinates of R by a factor of $\frac{1}{2}$, the area scales by the square of the factor:

$$\text{Area}(R') = \left(\frac{1}{2}\right)^2 \text{Area}(R) = \frac{1}{4}\text{Area}(R).$$

Given that Area$(R) > 4$, we have:

$$\text{Area}(R') > \frac{1}{4} \times 4 = 1.$$

Since Area$(R') > 1$, by Blichfeldt's Theorem, there exist distinct points $x', y' \in R'$ such that $x' - y' \in \mathbb{Z}^2 \setminus \{0\}$. This confirms there is a nonzero lattice point formed by the difference of two points in R'.

125 Let x' and y' be as in Exercise 124. Show that $x' - y'$ is in R. Hint: express $x' - y'$ as a linear combination of points that you know are in R.

By the definition of the set R', since $x', y' \in R'$, there exist points $x, y \in R$ such that $x' = \frac{1}{2}x$ and $y' = \frac{1}{2}y$.
We examine the difference $x' - y'$:

$$x' - y' = \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}(-y).$$

Since R is symmetric about the origin, if $y \in R$, then $-y \in R$.
Since R is a convex set, any convex combination of points in R must also be in R.
The expression $\frac{1}{2}x + \frac{1}{2}(-y)$ is a convex combination (specifically the midpoint) of the two points x and $-y$, both of which are in R. Therefore, their midpoint must lie in R:

$$\frac{1}{2}x + \frac{1}{2}(-y) \in R \implies x' - y' \in R.$$

This proves that there exists a nonzero lattice point inside the region R, which concludes the proof of Minkowski's Theorem.

Problem 16 The distribution of visible lattice points in the plane is $6/\pi^2$. Consider the square region $S(t)$ in the plane defined by $\lvert x\rvert \le t$ and $\lvert y\rvert \le t$. Let $N(t)$ denote the number of lattice points in this square, and let $V(t)$ denote the number of lattice points in the square that are visible from the origin. Show that $\lim_{t\to\infty} \frac{V(t)}{N(t)} = \frac{6}{\pi^2}$.

A lattice point (x, y) is visible from the origin if and only if $\gcd(x, y) = 1$. The total number of lattice points $N(t)$ in the region $[-t, t] \times [-t, t]$ is $(2\lfloor t\rfloor + 1)^2$. As $t \to \infty$, $N(t) \approx 4t^2$.
To find the number of visible points $V(t)$, we sum over all possible values of $d = \gcd(x, y)$. If $\gcd(x, y) = d$, then $x = da$ and $y = db$ where $\gcd(a, b) = 1$. Using the property of the Mobius function $\mu(d)$ where $\sum_{d\mid \gcd(x,y)} \mu(d) = 1$ if $\gcd(x, y) = 1$ and 0 otherwise, we can count visible points:

$$V(t) = \sum_{|x|\le t} \sum_{|y|\le t} \sum_{d\mid \gcd(x,y)} \mu(d).$$

Rearranging the sums, we let $x = da, y = db$:

$$V(t) = \sum_{d=1}^{\lfloor t\rfloor} \mu(d) \times (\text{number of lattice points in } [-t/d, t/d] \times [-t/d, t/d]).$$

The number of points in the scaled square is $(2\lfloor t/d\rfloor + 1)^2 \approx \frac{4t^2}{d^2}$.

$$V(t) \approx \sum_{d=1}^{\infty} \mu(d)\frac{4t^2}{d^2} = 4t^2 \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}.$$

Recalling that the sum $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$, for $s = 2$ we have:

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{1}{\pi^2/6} = \frac{6}{\pi^2}.$$

Thus, $V(t) \approx 4t^2 \cdot \frac{6}{\pi^2}$. Taking the limit of the ratio:

$$\lim_{t\to\infty} \frac{V(t)}{N(t)} = \frac{4t^2(6/\pi^2)}{4t^2} = \frac{6}{\pi^2}.$$

Reflection paragraph:
I think a particularly useful skill I have learned in geometry of numbers is to become incredibly familiar with the process of breaking problems down. Throughout the semester we have done a LOT of proofs, and very frequently, until proving all kinds of problems has become incredibly familiar to me. Breaking very complex problems down into simple parts used to be very difficult for me, but now I feel incredibly confident making any difficult problem solvable by just breaking it down into steps. These steps can be proofs by contradiction, possibilities for pigeonhole, minkowski's, etc., and it's really improved my academic life outside of mathematics as it has stretched into humanities classes and computer science problems.